

# Estimates for the Maxwell field near the spatial and null infinity of the Schwarzschild spacetime

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## Abstract

It is shown how the gauge of the “regular finite initial value problem at spacelike infinity” can be used to construct a certain type of estimates for the Maxwell field propagating on a Schwarzschild background. These estimates are constructed with the objective of obtaining information about the smoothness near spacelike and null infinity of a wide class of solutions to the Maxwell equations.

Keywords: General Relativity, asymptotic structure, Maxwell equations, spatial infinity.

## 1 Introduction

In reference [15], an analysis of the behaviour of the Maxwell field (spin-1 zero-rest-mass field) propagating near the spatial infinity of a Schwarzschild background was suggested as a way of gaining insight into certain aspects of the asymptotics of the gravitational field. Using Friedrich’s *cylinder at spatial infinity* representation of the region of the Schwarzschild spacetime which is “near” spatial infinity [2, 5] it was possible to discuss the occurrence of obstructions to the smoothness of the Maxwell field at null infinity. The analysis of these obstructions is done by means of a certain type of asymptotic expansions which can be calculated in the aforementioned formalism. The test Maxwell field propagating on the is obtained as the solution to an initial value problem with initial data prescribed on a  $t = \text{constant}$  slice of the conformally rescaled Schwarzschild spacetime. The most important aspect of the expansions is that they allow to relate in an explicit manner properties of the initial data with the behaviour of the field at null infinity. In particular, it was shown in [15] that these asymptotic expansions contain logarithmic divergences at the sets where spatial infinity “touches” null infinity —the so-called *critical sets*. A certain subset of the logarithmic divergences is still present if instead of propagation on a Schwarzschild background, one considers propagation on a flat background —hence, one can regard these logarithmic divergences as a structural property of the class of hyperbolic equations under consideration. This type of logarithmic divergences was first observed in the analysis of the conformal field equations carried out in [2]. As in the case of the conformal field equations, the analogous logarithmic singularities in the Maxwell field can be precluded by imposing certain regularity conditions on the initial data. The analysis in [15] shows that even if these regularity conditions are satisfied, there are some further logarithmic divergences which could be interpreted as arising from the interaction of the Maxwell field with the curved background. These logarithmic divergences are similar in structure to the ones observed in [12, 11, 13, 14].

Due to the hyperbolic nature of the Maxwell equations it is to be expected that the logarithmic divergences in the asymptotic expansions will propagate into null infinity, and hence will have an

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effect on the smoothness of the test field at the conformal boundary. In view of the results of [15], the challenge is to determine in a precise and rigorous manner how these properties of the asymptotic expansions translate into properties of the actual solutions to the Maxwell equations.

The fundamental structural properties of the type of evolution equations under consideration have been analysed at length elsewhere —see [2, 3, 4, 5, 10] in the case of the conformal field equations and [15] in the case of the Maxwell equations. The crucial observation message in these analyses is that although the relevant propagation equations happen to be symmetric-hyperbolic in the interior of the conformally rescaled spacetime, they degenerate at the critical sets in the sense that the matrix associated with the time derivative loses rank —this degeneracy is responsible, in particular, of the first class of logarithmic singularities discussed in previous paragraphs. As a consequence, the standard existence arguments for symmetric hyperbolic systems break down at the critical sets.

Gaining control on solutions of degenerate propagation equations requires an understanding of their *algebraic* properties —this is the rationale of the analysis in [15]. In a second stage, ideally, one has to learn how to relate these algebraic properties with estimates of the solutions to the equations. A first step in this direction has been given by Friedrich in [4] with the construction of a certain type of estimates for the spin-2 field on the Minkowski spacetime. This construction bypasses the technical problems mentioned in the previous paragraphs, and permits the use of information coming from asymptotic expansions.

The objective of the present work is to adapt the ideas of [4] to the case of the Maxwell field on a curved background —the Schwarzschild spacetime. The implementation these ideas is far from straightforward. Some of the problems one faces in this implementation were already foreseen in [4]: the construction of estimates was performed in a very particular conformal gauge, which in a sense, exploits to the maximum the simplicity of the Minkowski spacetime. The use of other gauges would include a string of extra terms which one has to learn how to control. As we shall see, the situation in a curved background is analogous: the curved background also introduces a string of lower order terms. The crucial idea in the construction of estimates for the spin-2 massless field on a flat background is that although at first sight it seems not possible to construct  $L^2$ -type estimates for the components of the field, it is nevertheless possible by means of an alternative argument to construct estimates for sufficiently high “radial” derivatives. These estimates, in turn, can be used to control the remainder of Taylor-like expansions of the Maxwell field. In order to extend these ideas to the case of a curved background, it will be necessary to take this insight a step further and assume that one has solutions to the Maxwell equations have a Taylor-like expansion —this last point was not assumed in [4].

This article is organised as follows: section 2 gives a brief review of Friedrich’s formalism of the cylinder at spatial infinity applied to the case of the Schwarzschild spacetime. This will be the setting of our discussion. Section 3 gives a discussion of relevant aspects of the Maxwell equations within the framework of the cylinder at spatial infinity. Section 4 contains a discussion of the construction of estimates for the Maxwell field on a Minkowski background. This section follows very closely a similar discussion for the spin-2 massless field given in [4], and is given here for completeness, reference and comparison with the discussion in subsequent sections. Section 5 gives the construction of estimates on a Schwarzschild background, and contains the main results of the article. Finally, there is a concluding section —section 6— containing a summary of the results obtained and listing all the assumptions being made. In addition there are two appendices: the first one, appendix A, listing the definitions of some spinorial objects used throughout the article, and a second one, appendix B, describing some crucial results concerning vector fields on the Lie algebra of  $SU(2)$ .

## 2 The Schwarzschild spacetime in the F-gauge

Some relevant aspects of the framework of the *cylinder at spatial infinity* or *F-gauge* for the case of the Schwarzschild spacetime are first discussed. This gauge allows to formulate, for the Einstein equations, a *regular finite initial value problem near spatial infinity*. The original construction

has been given in [2]. The Schwarzschild metric in isotropic coordinates is given by

$$\tilde{g} = \left( \frac{1 - m/2r}{1 + m/2r} \right)^2 dt^2 + \left( 1 + \frac{m}{2r} \right)^4 (dr^2 + r^2 d\sigma^2).$$

Consider the time-symmetric hypersurface  $\tilde{\mathcal{S}}$ ,  $t = \text{constant}$  and let  $\tilde{h}$  denote its (negative definite) intrinsic metric. Writing  $\tilde{h} = \Omega^{-2}h$ , one finds the following conformal intrinsic metric and conformal factor:

$$h = -(d\rho^2 + \rho^2 d\sigma^2), \quad \Omega = \frac{\rho^2}{(1 + \rho m/2)^2}, \quad (1)$$

where the radial coordinate  $\rho = 1/r$  has been introduced. Let  $i_1$  and  $i_2$  denote the infinities corresponding to the two asymptotic ends of the hypersurface  $\tilde{\mathcal{S}}$ . Further, let  $\mathcal{S} = \tilde{\mathcal{S}} \cup \{i_1, i_2\}$ . Let  $i = i_1$  denote the infinity corresponding to the locus  $\rho = 0$ . The discussion in this article will be concerned with the domain of influence,  $J^+(\mathcal{B}_{a_*}(i))$  of a sufficiently small ball,  $\mathcal{B}_a(i)$ , of radius  $a_*$  based on  $i$ . The point  $i$  can be blown up to a 2-sphere,  $\mathcal{S}^2$ . Accordingly, introduce the set  $\mathcal{C}_{a_*} = (\mathcal{B}_{a_*}(i) \setminus i) \cup \mathcal{S}^2$ .

The use of a gauge based on *conformal Gaussian coordinates* leads to a spacetime conformal factor given by

$$\Theta = \frac{\Omega}{\kappa} \left( 1 - \tau^2 \frac{\kappa^2}{\omega^2} \right), \quad (2)$$

where

$$\omega = \frac{2\Omega}{\sqrt{|D^i \Omega D_i \Omega|}} = \rho(1 + \rho m/2), \quad (3)$$

and  $\kappa > 0$  is a smooth function such that  $\kappa = \mu\rho$  with  $\mu(i) = 1$ ;  $D_i$  denotes the Levi-Civita connection of the (flat) metric  $h$ . The function  $\kappa$  encodes the remaining conformal freedom in the setting. In order to avoid disrupting the spherical symmetry of the representation, the analysis will be restricted to spherically symmetric choices of  $\kappa$ . Usual choices for  $\kappa$  are

$$\kappa = \rho, \quad (4a)$$

$$\kappa = \omega. \quad (4b)$$

*For the purposes of the current article, it turns out that the choice of  $\kappa$  given by (4b) is more convenient.*

The coordinate  $\tau$  in (2) is an affine parameter of conformal geodesics whose tangent at  $\tau = 0$  is parallel to the normal of  $\mathcal{S}$ . Using these conformal geodesics one constructs *conformal Gaussian coordinates*: the coordinate  $\rho$  can be extended off  $\mathcal{S}$  by requiring it to be constant along the aforementioned conformal geodesics. ‘‘Angular coordinates’’ can be extended in a similar fashion.

In view of the conformal factor (2) define the manifold

$$\mathcal{M}_{a_*, \kappa} = \left\{ (\tau, q) \mid q \in \mathcal{C}_{a_*}, -\frac{\omega}{\kappa} \leq \tau \leq \frac{\omega}{\kappa} \right\},$$

and the following relevant subsets thereof:

$$\begin{aligned} \mathcal{I} &= \left\{ (\tau, q) \in \mathcal{M}_{a_*, \kappa} \mid \rho(q) = 0, |\tau| < \frac{\omega}{\kappa} \right\}, \\ \mathcal{I}^\pm &= \left\{ (\tau, q) \in \mathcal{M}_{a_*, \kappa} \mid \rho(q) = 0, \tau = \pm 1 \right\}, \\ \mathcal{I}^0 &= \left\{ (\tau, q) \in \mathcal{M}_{a_*, \kappa} \mid \rho = 0, \tau = 0 \right\}, \\ \mathcal{J}^\pm &= \left\{ (\tau, q) \in \mathcal{M}_{a_*, \kappa} \mid q \in \mathcal{B}_{a_*}(i) \setminus \mathcal{I}^0, \tau = \pm \frac{\omega}{\kappa} \right\}, \end{aligned}$$

denoting, respectively, the *cylinder at spatial infinity*, the *critical sets* where spatial infinity touches null infinity, the intersection of the cylinder at spatial infinity with the initial hypersurface  $\mathcal{S}$ , and the two components of *null infinity*. In particular, with the choice (4b) of  $\kappa$ , the locus of null infinity is given by  $\tau = \pm 1$ . The manifold  $\mathcal{M}_{a_*, \kappa}$  with the gauge choice  $\kappa = \omega$ , will be denoted by  $\mathcal{M}_{a_*, \omega}$ .

It will be convenient to work with a space-spinor formalism —see e.g. [7]. In order to write down the field equations, introduce a null frame  $c_{AA'}$  satisfying  $g(c_{AA'}, c_{BB'}) = \epsilon_{AB}\epsilon_{A'B'}$ . Let  $\tau_{AA'}$  —with normalisation  $\tau^{AA'}\tau_{AA'} = 2$ — be tangent to the conformal geodesics of which  $\tau$  is a parameter. The frame can be split into

$$c_{AA'} = \frac{1}{2}\tau_{AA'}\tau^{CC'}c_{CC'} - \tau^B_{A'}c_{AB},$$

with

$$\tau^{AA'}c_{AA'} = \sqrt{2}\partial_\tau, \quad c_{AB} = \tau_{(A}{}^{B'}c_{B)B'}.$$

In particular, the following choice will be made:

$$c_{00'} = \frac{1}{\sqrt{2}}((1+c^0)\partial_\tau + c^1\partial_\rho), \quad c_{11'} = \frac{1}{\sqrt{2}}((1-c^0)\partial_\tau - c^1\partial_\rho),$$

with  $c^0$  and  $c^1$  functions of  $(\tau, \rho)$ . The remaining vectors of the frame,  $c_{01'}$  and  $c_{10'}$  must then be tangent to the spheres  $\{\tau = \text{constant}, \rho = \text{constant}\}$ , and thus cannot define smooth vector fields everywhere. To avoid this difficulty all possible tangent vectors  $c_{01'}$  and  $c_{10'}$  will be considered. This results in a 5-dimensional submanifold of the bundle of normalised spin frames. Rotations  $c_{01'} \rightarrow e^{i\vartheta}c_{01'}$ ,  $\vartheta \in \mathbb{R}$  leave this submanifold invariant. Hence it defines a subbundle with structure group  $U(1)$  which projects into  $\mathcal{M}_{a*,\omega}$ . All the relevant structures will be lifted to the subbundle, which in an abuse of notation will be again denoted by  $\mathcal{M}_{a*,\omega}$  and which is diffeomorphic to  $[-1, 1] \times [0, \infty) \times SU(2)$ .

The introduction of the bundle space  $\mathcal{M}_{a*,\omega}$  and of the frame  $c_{AA'}$  in our formalism implies that all relevant quantities have a definite spin weight and hence admit an expansion in terms of some functions  $T_j^k$  associated with unitary representations of  $SU(2)$  —see e.g. [2, 5] for a more detailed discussion in this respect. One can introduce differential operators  $X$ ,  $X_+$  and  $X_-$  defined by their action on the functions  $T_j^k$ . With the help of these operators one can write

$$c_{AA'} = c_{AA'}^\mu \partial_\mu = c_{AA'}^0 \partial_\tau + c_{AA'}^1 \partial_\rho + c_{AA'}^+ X_+ + c_{AA'}^- X_-.$$

In addition to the frame  $c_{AA'}$ , in the F-gauge the geometry of  $\mathcal{M}_{a,\kappa}$  is described by means of the associated connection  $\Gamma_{AA'BC}$ , the spinorial counterpart of the Ricci tensor of a Weyl connection,  $\Theta_{AA'BB'}$ , and the rescaled Weyl spinor,  $\phi_{ABCD}$ . Its unprimed (i.e. space-spinor) version of the connection spinor is given by  $\Gamma_{ABCD} = \tau_B{}^{B'}\Gamma_{AB'CD}$ ,  $\Gamma_{AA'CD} = \Gamma_{ABCD}\tau^B{}_{A'}$ , which is decomposed as

$$\Gamma_{ABCD} = \frac{1}{\sqrt{2}}(\xi_{ABCD} - \chi_{(AB)CD}) - \frac{1}{2}\epsilon_{AB}f_{CD}.$$

Similarly, one considers  $\Theta_{ABCD} = \tau_C{}^{A'}\tau_D{}^{B'}\Theta_{AA'BB'}$ . The explicit spherical symmetry of the spacetime justifies the following Ansatz in terms of irreducible spinors:

$$\begin{aligned} c_{AB}^0 &= c^0 x_{AB}, & c_{AB}^1 &= c^1 x_{AB}, & c_{AB}^- &= c^- y_{AB}, & c_{AB}^+ &= c^+ z_{AB}, \\ f_{AB} &= f x_{AB}, & \xi_{ABCD} &= \xi(\epsilon_{AC}x_{BD} + \epsilon_{BD}x_{AC}), \\ \chi_{(AB)CD} &= \chi_2 \epsilon_{ABCD}^2 + \chi_h h_{ABCD}, \\ \Theta_{ABCD} &= \Theta_2 \epsilon_{ABCD}^2 + \Theta_h h_{ABCD} + \Theta_x \epsilon_{AB}x_{CD}, & \phi_{ABCD} &= \phi \epsilon_{ABCD}^2. \end{aligned}$$

The definitions of the irreducible spinors introduced above is given in the appendix. The manifest spherical symmetry of this representation implies that the functions  $c^0$ ,  $c^1$ ,  $c^\pm$ ,  $f$ ,  $\xi$ ,  $\chi_2$ ,  $\chi_h$ ,  $\phi$  have spin-weight 0. Furthermore they only contain the function  $T_0^0 = 1$ .

The functions  $c^0$ ,  $c^1$ ,  $c^\pm$ ,  $f$ ,  $\xi$ ,  $\chi_2$ ,  $\chi_h$ ,  $\phi$  are determined by solving the conformal propagation equations discussed in [2] with the appropriate initial data. The problem of reconstructing the conformal Schwarzschild solution from the given data amounts to finding a solution  $u = u(\tau, \rho; m)$  of an initial value problem of the type

$$\partial_\tau u = F(u, \tau, \rho; m), \quad u(0, \rho; m) = u_0(\rho; m), \quad (7)$$

with analytic functions  $F$  and  $u_0$ . The solution with  $m = 0$  corresponds to a portion of the conformal Minkowski spacetime, in which the only non-vanishing components of the solution are given by

$$c^0 = -\mu\tau, \quad c^1 = \kappa, \quad c^\pm = \mu, \quad f = \kappa',$$

where  $'$  denotes differentiation with respect to  $\rho$  and  $\kappa = \mu\rho$ . Since in this case the solution exists for all  $\tau, \rho \in \mathbb{R}$ , it can be shown that for a given  $m$  there is sufficiently small  $\rho_0$  such that there is an analytic solution to the system (7) which extends beyond  $\mathcal{I}$  for  $\rho < \rho_0 < a_*$ . Hence, if  $a_*$  is taken to be small enough, one can recover the portion of the Schwarzschild spacetime near null and spatial infinity.

It follows from the above discussion that the coefficients that are obtained from solving the transport propagation equations on the cylinder at spatial infinity—as discussed in [2, 12]—correspond to the first terms in the expansions of the solutions of (7).

### Information about null infinity

In the sequel, it will be necessary to have some more precise information about the behaviour of the frame spinors  $c_{AB}$ . One of the remarkable features of the present conformal setting is that it allows to obtain information about some field quantities at null infinity without the need of explicitly solving the field equations. In [1] it has been shown that for the conformal Gaussian coordinates the following relation holds:

$$\Theta f_{AB} = d_{AB} - c_{AB}(\Theta), \quad (8)$$

where  $d_{AB}$  is a space spinor associated to the conformal factor  $\Theta$ , but independent of the choice of  $\kappa$ —see [2]. It is determined entirely by the initial data on  $\mathcal{S}$ . In the case of the Schwarzschild spacetime one has

$$d_{AB} = \frac{2\rho x_{AB}}{(1 + \rho m/2)}. \quad (9)$$

Thus, if  $f_{AB}$ ,  $c_{AB}$  and  $d_{AB}$  are smooth at points where  $\Theta = 0$ —as it is the case on null infinity—then from (8) it follows that

$$d_{AB} = c_{AB}(\Theta).$$

Now, in particular

$$c_{01}(\Theta) = \frac{1}{\sqrt{2}} (c^0 \partial_\tau \Theta + c^1 \partial_\rho \Theta).$$

For the choice  $\kappa = \omega$  the conformal factor (2) takes the form  $\Theta = \Omega\omega^{-1}(1 - \tau^2)$ , so that one computes

$$c_{01}(\Theta)|_{\mathcal{I}^+} = -\sqrt{2}\frac{\Omega}{\omega}c^0|_{\mathcal{I}^+}.$$

Hence, using (1), (3) and (9) one concludes that

$$c^0|_{\mathcal{I}^+} = -1, \quad (10)$$

if  $\kappa = \omega$ . An analogous calculation for  $\mathcal{I}^-$  renders

$$c^0|_{\mathcal{I}^-} = 1. \quad (11)$$

Consequences of these results with regards to the Maxwell equations will be discussed in the following section.

## 3 The Maxwell field

The Maxwell field will be described by means of totally symmetric valence 2 spinor  $\phi_{AB}$ , the *Maxwell spinor*, related to the spinorial counterpart of the Maxwell tensor via

$$F_{AA'BB'} = \phi_{AB}\epsilon_{A'B'} + \bar{\phi}_{A'B'}\epsilon_{AB}.$$

The Maxwell equations are equivalent to the *spin-1 zero-rest mass field equations*:

$$\nabla^{AA'} \phi_{AB} = 0.$$

If the conformal weight of  $\phi_{AB}$  is chosen properly, the vacuum Maxwell equations are conformally invariant. If  $\tilde{\phi}_{AB}$  denotes the *physical* Maxwell spinor, then the Maxwell spinor in the conformally rescaled (unphysical) spacetime is given by

$$\phi_{AB} = \Theta^{-1} \tilde{\phi}_{AB}.$$

Due to the totally symmetric character of  $\phi_{AB}$ , one can write

$$\phi_{AB} = \phi_0 \epsilon_{AB}^0 + \phi_1 \epsilon_{AB}^1 + \phi_2 \epsilon_{AB}^2,$$

where the totally symmetric spinors  $\epsilon_{AB}^0$ ,  $\epsilon_{AB}^1$  and  $\epsilon_{AB}^2$  are defined in appendix A. In the F-gauge, and using a space-spinor decomposition, the Maxwell equations can be shown to imply the following system of 4 equations:

$$\begin{aligned} (\sqrt{2} - 2c_{01}^0) \partial_\tau \phi_0 + 2c_{00}^0 \partial_\tau \phi_1 - 2c_{01}^\alpha \partial_\alpha \phi_0 + 2c_{00}^\alpha \partial_\alpha \phi_1 \\ = (2\Gamma_{0011} - 4\Gamma_{1010}) \phi_0 + 4\Gamma_{1000} \phi_1 - 2\Gamma_{0000} \phi_2, \end{aligned} \quad (12a)$$

$$\begin{aligned} (\sqrt{2} - 2c_{01}^0) \partial_\tau \phi_1 + 2c_{00}^0 \partial_\tau \phi_2 - 2c_{01}^\alpha \partial_\alpha \phi_1 + 2c_{00}^\alpha \partial_\alpha \phi_2 \\ = -2\Gamma_{0111} \phi_0 + 4\Gamma_{0011} \phi_1 - (3\Gamma_{0001} + 2\Gamma_{0100}) \phi_2, \end{aligned} \quad (12b)$$

$$\begin{aligned} (\sqrt{2} + 2c_{01}^0) \partial_\tau \phi_1 - 2c_{11}^0 \partial_\tau \phi_0 + 2c_{01}^\alpha \partial_\alpha \phi_1 - 2c_{11}^\alpha \partial_\alpha \phi_0 \\ = (2\Gamma_{0111} - 4\Gamma_{1110}) \phi_0 + 4\Gamma_{1100} \phi_1 + (2\Gamma_{0100} - 3\Gamma_{0001}) \phi_2, \end{aligned} \quad (12c)$$

$$\begin{aligned} (\sqrt{2} + 2c_{01}^0) \partial_\tau \phi_2 - 2c_{11}^0 \partial_\tau \phi_1 + 2c_{01}^\alpha \partial_\alpha \phi_2 - 2c_{11}^\alpha \partial_\alpha \phi_1 \\ = -2\Gamma_{1111} \phi_0 + 4\Gamma_{0111} \phi_1 + (2\Gamma_{1100} - 4\Gamma_{0101}) \phi_2. \end{aligned} \quad (12d)$$

This system of equations resembles in its form the Maxwell equations in the Newman-Penrose formalism —see for example [8]. Taking linear combinations of equations (12b) and (12c) one recovers the system of *propagation equations* and the *constraint equation* given in [15]. Particularising to the case of a Schwarzschild background one obtains

$$\begin{aligned} (1 - c^0) \partial_\tau \phi_0 - c^1 \partial_\rho \phi_0 + c^+ X_+ \phi_1 - \Gamma_0 \phi_0 &= 0, \\ (1 - c^0) \partial_\tau \phi_1 - c^1 \partial_\rho \phi_1 + c^+ X_+ \phi_2 - \Gamma_1 \phi_1 &= 0, \\ (1 + c^0) \partial_\tau \phi_1 + c^1 \partial_\rho \phi_1 + c^- X_- \phi_0 - \Xi_1 \phi_1 &= 0, \\ (1 + c^0) \partial_\tau \phi_2 + c^1 \partial_\rho \phi_2 + c^- X_- \phi_1 - \Xi_2 \phi_2 &= 0. \end{aligned}$$

where  $c^0$ ,  $c^1$ ,  $c^\pm$  are the analytic functions in the coordinates  $(\tau, \rho)$  solving the system (7) discussed in section 2. The functions  $\Gamma_0$ ,  $\Gamma_1$ ,  $\Xi_1$  and  $\Xi_2$  are linear combinations of the analytic *components of the connection*  $f$ ,  $\xi$ ,  $\chi_2$  and  $\chi_h$ . More precisely,

$$\begin{aligned} \Gamma_0 &= 2\Gamma_{0011} - 4\Gamma_{1010} = 2\xi + \frac{\sqrt{2}}{6} \chi_2 - 2\sqrt{2} \chi_h - \sqrt{2} f, \\ \Gamma_1 &= 4\Gamma_{0011} = 3\xi - \frac{5\sqrt{2}}{12} \chi_2 - \frac{5\sqrt{2}}{2} \chi_h, \\ \Xi_1 &= -4\Gamma_{1100} = 3\xi - \frac{\sqrt{2}}{4} \chi_2 + \frac{3\sqrt{2}}{2} \chi_h, \\ \Xi_2 &= 2\Gamma_{1100} - 4\Gamma_{0101} = -2\xi + \frac{\sqrt{2}}{6} \chi_2 - 2\sqrt{2} \chi_h + \sqrt{2} f. \end{aligned}$$

The precise form of these functions will not be required in our discussion —a list of the leading terms of their asymptotic expansions in the gauge for which  $\kappa = \rho$  can be found in the appendix of [15]. In the sequel, it will be convenient to isolate the leading terms of these functions. We write:

$$\begin{aligned} c^0 &= -\tau - a, \quad c^1 = \rho + b, \\ c^+ &= 1 + c, \quad c^- = 1 + c, \\ \Gamma_0 &= -1 - f_0, \quad \Gamma_1 = -f_1, \\ \Xi_1 &= -g_0, \quad \Xi_2 = 1 - g_1, \end{aligned}$$

where  $a, b, c, f_0, f_1, g_0, g_1$  are analytic functions of  $\tau, \rho$  independent of the “angular coordinates”  $\varsigma \in SU(2)$ . One has that

$$\begin{aligned} a &= \mathcal{O}(\rho), & b &= \mathcal{O}(\rho^2), & c &= \mathcal{O}(\rho), \\ f_0 &= \mathcal{O}(\rho), & f_1 &= \mathcal{O}(\rho), & g_0 &= \mathcal{O}(\rho), & g_1 &= \mathcal{O}(\rho). \end{aligned}$$

Further, they all vanish if  $m = 0$  where  $m$  is the mass of the Schwarzschild spacetime. Due to their monopolar nature

$$\begin{aligned} X_{\pm}a &= X_{\pm}b = X_{\pm}c = X_{\pm}f_0 = X_{\pm}f_1 = X_{\pm}g_0 = X_{\pm}g_1 = 0, \\ Xa &= Xb = Xc = Xf_0 = Xf_1 = Xg_0 = Xg_1 = 0, \end{aligned}$$

where  $X_+, X_-$  and  $X$  are the differential operators on  $SU(2)$  discussed in section 2. Using this notation, the equations take the form

$$A_0 \equiv (1 + \tau + a)\partial_{\tau}\phi_0 - (\rho + b)\partial_{\rho}\phi_0 + (1 + c)X_+\phi_1 + (1 + f_0)\phi_0 = 0, \quad (15a)$$

$$A_1 \equiv (1 + \tau + a)\partial_{\tau}\phi_1 - (\rho + b)\partial_{\rho}\phi_1 + (1 + c)X_+\phi_2 + f_1\phi_1 = 0, \quad (15b)$$

$$B_1 \equiv (1 - \tau - a)\partial_{\tau}\phi_1 + (\rho + b)\partial_{\rho}\phi_1 + (1 + c)X_-\phi_0 + g_0\phi_1 = 0, \quad (15c)$$

$$B_2 \equiv (1 - \tau - a)\partial_{\tau}\phi_2 + (\rho + b)\partial_{\rho}\phi_2 + (1 + c)X_-\phi_1 - (1 - g_1)\phi_2 = 0, \quad (15d)$$

which is the form that will be used in the rest of the article.

### On the characteristics of the Maxwell equations in the F-gauge

In the following, certain questions concerning the characteristics of the equations (15a)-(15d). From general theory, these have to coincide with null hypersurfaces in  $\mathcal{M}_{a_*, \omega}$  and can be grouped in *outgoing* and *incoming* according to whether they intersect future or past null infinity. Let  $(\tilde{\tau}(s), \tilde{\rho}(s))$  be the solutions to the system of ordinary differential equations

$$\begin{aligned} \frac{d\tilde{\tau}(s)}{ds} &= \left(1 + \tilde{\tau}(s) + a(\tilde{\tau}(s), \tilde{\rho}(s))\right), & \tilde{\tau}(0) &= 0, \\ \frac{d\tilde{\rho}(s)}{ds} &= -\left(\tilde{\rho}(s) + b(\tilde{\tau}(s), \tilde{\rho}(s))\right), & \tilde{\rho}(0) &= \rho_{\bullet}, \end{aligned}$$

for  $0 \leq \rho_{\bullet} < a_*$ . Similarly, let  $(\hat{\tau}(s), \hat{\rho}(s))$  be the solutions of

$$\begin{aligned} \frac{d\hat{\tau}(s)}{ds} &= \left(1 - \hat{\tau}(s) - a(\hat{\tau}(s), \hat{\rho}(s))\right), & \hat{\tau}(0) &= 0, \\ \frac{d\hat{\rho}(s)}{ds} &= \left(\hat{\rho}(s) + b(\hat{\tau}(s), \hat{\rho}(s))\right), & \hat{\rho}(0) &= \rho_{\bullet}. \end{aligned}$$

In the case of Minkowski spacetime, the solutions to the above equations as given —after a change of parameter— by

$$\begin{aligned} \tilde{\tau}(s') &= \frac{s'}{1 - s'}, & \tilde{\rho}(s') &= \rho_{\bullet}(1 - s'), \\ \hat{\tau}(s') &= \frac{s'}{1 + s'}, & \hat{\rho}(s') &= \rho_{\bullet}(1 + s'), \end{aligned}$$

for  $s' \in [-1/2, 1/2]$ . As in the case of the Minkowski spacetime, it can be shown that for  $(\tilde{\tau}(s), \tilde{\rho}(s))$  and  $\rho_{\bullet} \neq 0$ , there is a  $s_{\mathcal{J}^+} \in \mathbb{R}$  such that  $\tilde{\tau}(s_{\mathcal{J}^+}) = 1$  and  $\tilde{\rho}(s_{\mathcal{J}^+}) \neq 0$ . On the other hand one can show that  $\hat{\tau}(s) \neq 1$  for all  $s$  —in fact,  $\hat{\tau}(s) \rightarrow 1$  as  $s \rightarrow \infty$ . Analogous statements can be made for past null infinity. In accordance with the previous discussion

$$\begin{aligned} \tilde{\mathcal{B}}_{\rho_{\bullet}} &= \left\{ (\tau, \rho, \varsigma) \in \mathcal{M}_{a, \omega} \left| \tau = \tilde{\tau}(s), \rho = \tilde{\rho}(s), s \in (s_-, s_+) \right. \right\}, \\ \hat{\mathcal{B}}_{\rho_{\bullet}} &= \left\{ (\tau, \rho, \varsigma) \in \mathcal{M}_{a, \omega} \left| \tau = \hat{\tau}(s), \rho = \hat{\rho}(s), s \in (s_-, s_+) \right. \right\}, \end{aligned}$$

correspond, respectively, to *outgoing* and *incoming* characteristics of (15a)-(15d). As discussed in [2] as  $\rho_0 \rightarrow 0$  the sets  $\tilde{\mathcal{B}}_{\rho_\bullet}$  and  $\tilde{\mathcal{B}}_{\rho_\bullet}$  approach  $\mathcal{I} \cup \mathcal{J}^-$  and  $\mathcal{I} \cup \mathcal{J}^+$ , respectively, in a non-uniform manner. In this sense, the set  $\mathcal{I}$  can be regarded as a limit set of both incoming and outgoing geodesics—a *total characteristic*.

### On the degeneracy of the propagation equations at the conformal boundary

As discussed in [2, 4, 15] a structural property of propagation equations derived from a covariant equation with principal part of the form  $\nabla^{AA'} \phi_{A\dots P}$ —as in the case of the propagation equations (15a)-(15d)—is the degeneracy of subsets of them at the critical sets  $\mathcal{I}^\pm$ . This can be readily seen in equations (15a)-(15d) by noting that  $a|_{\mathcal{I}^\pm} = 0$  and hence

$$(1 + \tau + a)|_{\mathcal{I}^+} = 0, \quad (1 - \tau - a)|_{\mathcal{I}^-} = 0.$$

This state of affairs can be “worsened” by some choices of conformal gauge. In particular if the function  $\kappa$  in the conformal factor (2) is chosen such that  $\kappa = \omega$ —as it is done in the present calculations—then using (10) and (11) one has that

$$(1 + \tau + a)|_{\mathcal{J}^+} = 0, \quad (1 - \tau - a)|_{\mathcal{J}^-} = 0.$$

It turns out that this “worsening” of the degeneracy of the propagation equations at the conformal boundary does not complicate the analysis any further, and actually makes some of the calculations easier.

### Asymptotic expansions

As discussed in [15] the structural properties of the equations (15a)-(15d) allow to calculate asymptotic expansions of the components  $\phi_0$ ,  $\phi_1$  and  $\phi_2$  of the form

$$\phi_k \sim \sum_{l \geq |1-k|} \frac{1}{l!} \phi_k^{(l)} \rho^l, \quad \phi_k^{(l)} = \partial_\rho^l \phi_k|_{\rho=0}, \quad (16)$$

for  $k = 0, 1, 2$ . The coefficients in the expansion (16) are determined by exploiting the fact that the cylinder at infinity  $\mathcal{I}$  is a total characteristic of equations (15a)-(15d): the equations reduce to a system of interior equations when evaluated on  $\mathcal{I}$ . To exploit this feature it shall be assumed that the initial data for the equations (15a)-(15d) on  $\mathcal{C}_{a_*}$  are of the form

$$\phi_j = \sum_{l=|1-j|}^{\infty} \sum_{q=|1-j|}^l \sum_{k=0}^{2q} \frac{1}{l!} w_{j,l;q,k} T_{2q}^k \rho_{q+j-1}^l, \quad (17)$$

for  $j = 0, 1, 2$  with  $w_{j,l;q,k} \in \mathbb{C}$ . The initial data is subject to the constraint

$$\rho \partial_\rho \phi_1 + \frac{1}{2} X_- \phi_0 - \frac{1}{2} X_+ \phi_2 = 0,$$

so that all the coefficients in the expansions of  $\phi_1$ —except for  $w_{1,0;0,0}$ , the electric charge—are determined from those of  $\phi_0$  and  $\phi_2$ .

Using initial data of the form (17) it is possible to calculate the asymptotic expansions (16)—that is, the  $(\tau, \varsigma)$ -dependent coefficients  $\phi_k^{(l)}$ ,  $\varsigma \in SU(2)$ —to any desired order—the only limitation being the computational complexities. This procedure is a way of unfolding the evolution process so that it can be analysed in detail and to any order. In particular, the expansions allow to relate properties of the initial data with behaviour at null infinity.

The properties of the coefficients  $\phi_k^{(l)}$  and the occurrence of logarithmic singularities in them at the critical sets  $\mathcal{I}^\pm$ , has been the topic of [15]. Here, we start to analyse the way in which the expansions (16) are related to actual solutions of (15a)-(15d). To this end it will be assumed that given an integer  $p > 0$  the solutions of the (15a)-(15d) are of the form

$$\phi_k = \sum_{l=|1-k|}^{p-1} \frac{1}{l!} \phi_k^{(l)} \rho^l + R_p(\phi_k), \quad \phi_k^{(l)} = \partial_\rho^l \phi_k|_{\rho=0}, \quad (18)$$



with the residue of order  $p$  of  $\phi_k$ ,  $R_p(\phi_k)$ , given by

$$R_p(\phi_k) = J^p(\partial_\rho^p \phi_k),$$

where  $J$  denotes the operator  $f \mapsto J(f)$  such that

$$J(f)(\rho) = \int_0^\rho f(s) ds.$$

The objective of this article is to obtain estimates on the solutions to (15a)-(15d) by exploiting the Ansatz (18). As the coefficients  $\phi_k^{(l)}$ , for  $0 \leq l \leq p$  and  $k = 0, 1, 2$  are in principle known explicitly—and hence also their regularity—the latter implies obtaining estimates on  $\partial_\rho^p \phi_k$ .

*In what follows it will be assumed that the coefficients  $\phi_k^{(l)}$  solution of the Maxwell transport equations on  $\mathcal{I}$  are as smooth as necessary for our calculations to make sense.* This assumption implies, in turn, a restriction on the class of initial data on  $\mathcal{C}_{a_*}$  to be considered—that is, the coefficients in the expansions (17) for  $j = 0, 2$  should satisfy some regularity conditions. The discussion in [15] shows how this can be done.

## 4 Construction of estimates in Minkowski spacetime

As pointed out in [4] for the case of the spin-2 field on flat spacetime, the degeneracy of the propagation equations at the critical sets  $\mathcal{I}^\pm$  has as consequence that estimates obtained using the standard argument for symmetric hyperbolic equations—see e.g. [6, 9]—are not bounded at  $\mathcal{I}^\pm$ , and depending on the precise details of the conformal gauge, possibly also not bounded at  $\mathcal{J}^\pm$ . In [4] it was possible to overcome this difficulty by noticing that although it is not possible to obtain directly estimates of  $\phi_k$ , it is possible to obtain estimates for  $\partial_\rho^p \phi_k$ . This information can be used to control the residue in the expansion (18).

The methods in [4] can be readily transcribed for the case of the Maxwell field. For completeness we present a summary of the arguments. It will serve to motivate the discussion for the curved spacetime case and also to highlight the new complications that arise.

In the case of the Minkowski spacetime and with the choice of conformal gauge given by  $\kappa = \omega = \rho$ , equations (15a)-(15d) take the form

$$A_k \equiv (1 + \tau)\partial_\tau \phi_k - \rho\partial_\rho \phi_k + X_+ \phi_{k+1} + (1 - k)\phi_k, \quad (19a)$$

$$B_k \equiv (1 - \tau)\partial_\tau \phi_{k+1} + \rho\partial_\rho \phi_{k+1} + X_- \phi_k - k\phi_{k+1}, \quad (19b)$$

with  $k = 0, 1$ . As it is usual in the construction of estimates, it is assumed that one has a solution  $\phi_k$ ,  $k = 0, 1, 2$  of the required smoothness. Consider

$$D^{q,p,\alpha} \bar{\phi}_k D^{q,p,\alpha} A_k + D^{q,p,\alpha} \phi_k D^{q,p,\alpha} \bar{A}_k + D^{q,p,\alpha} \bar{\phi}_{k+1} D^{q,p,\alpha} B_k + D^{q,p,\alpha} \phi_{k+1} D^{q,p,\alpha} \bar{B}_k = 0, \quad (20)$$

for  $k = 0, 1$ . The notation

$$D^{q,p,\alpha} = \partial_\tau^q \partial_\rho^p Z^\alpha = \partial_\tau^q \partial_\rho^p Z_1^{\alpha_1} Z_2^{\alpha_2} Z_3^{\alpha_3}, \quad \alpha = (\alpha_1, \alpha_2, \alpha_3), \quad |\alpha| = \alpha_1 + \alpha_2 + \alpha_3,$$

has been introduced and will be used throughout the rest of the article. The operators  $Z_1$ ,  $Z_2$  and  $Z_3$  denote the differential operators over  $SU(2)$  discussed in appendix B. These operators can be written as linear combinations of the operators  $X_\pm$  and  $X$  discussed in section (2).

The expression (20) can be rewritten as

$$\begin{aligned} 0 = & \partial_\tau \left( (1 + \tau) |D^{q,p,\alpha} \phi_k|^2 + (1 - \tau) |D^{q,p,\alpha} \phi_{k+1}|^2 \right) + \partial_\rho \left( \rho |D^{q,p,\alpha} \phi_{k+1}|^2 - \rho |D^{q,p,\alpha} \phi_k|^2 \right) \\ & + \left( D^{q,p,\alpha} \bar{\phi}_k D^{q,p,\alpha} X_+ \phi_{k+1} + D^{q,p,\alpha} \phi_{k+1} D^{q,p,\alpha} X_+ \bar{\phi}_k \right) \\ & + \left( D^{q,p,\alpha} \phi_k D^{q,p,\alpha} X_- \bar{\phi}_{k+1} + D^{q,p,\alpha} \bar{\phi}_{k+1} D^{q,p,\alpha} X_- \phi_k \right) \\ & - 2(p - q + k - 1) |D^{q,p,\alpha} \phi_k|^2 + 2(p - q - k) |D^{q,p,\alpha} \phi_{k+1}|^2. \end{aligned} \quad (21)$$

For  $t \in [0, 1]$  and  $0 < \rho_* < a_*$ , introduce the following subsets of  $\mathcal{M}_{a_*, \omega} \approx [-1, 1] \times [0, \infty) \times SU(2)$ :

$$\mathcal{N}_t = \left\{ (\tau, \rho, \varsigma) \in \mathcal{M}_{a_*, \omega} \left| 0 \leq \tau \leq t, 0 \leq \rho \leq \frac{\rho_*}{1 + \tau} \right. \right\}, \quad (22a)$$

$$\mathcal{B}_t = \left\{ (\tau, \rho, \varsigma) \in \mathcal{M}_{a_*, \omega} \left| 0 \leq \tau \leq t, \rho = \frac{\rho_*}{1 + \tau} \right. \right\}, \quad (22b)$$

$$\mathcal{S}_t = \left\{ (\tau, \rho, \varsigma) \in \mathcal{M}_{a_*, \omega} \left| \tau = t, 0 \leq \rho \leq \frac{\rho_*}{1 + \tau} \right. \right\}, \quad (22c)$$

$$\mathcal{I}_t = \left\{ (\tau, \rho, \varsigma) \in \mathcal{M}_{a_*, \omega} \left| 0 \leq \tau \leq t, \rho = 0 \right. \right\}. \quad (22d)$$

Hence  $\mathcal{N}_t$  is the domain of influence of  $\mathcal{S}_0 \subset \mathcal{C}_{a_*}$ . Let  $d\mu$  denote the Haar measure over  $SU(2)$ . Integrating expression (21) over  $\mathcal{N}_t$  and noting that because of Gauss' theorem

$$\begin{aligned} & \int_{\mathcal{N}_t} \left( \partial_\tau \left( (1 + \tau) |D^{q,p,\alpha} \phi_k|^2 + (1 - \tau) |D^{q,p,\alpha} \phi_{k+1}|^2 \right) + \partial_\rho \left( \rho |D^{q,p,\alpha} \phi_{k+1}|^2 - \rho |D^{q,p,\alpha} \phi_k|^2 \right) \right) d\tau d\rho d\mu \\ &= \int_{\mathcal{S}_t} \left( (1 + t) |D^{q,p,\alpha} \phi_k|^2 + (1 - t) |D^{q,p,\alpha} \phi_{k+1}|^2 \right) d\rho d\mu - \int_{\mathcal{S}_0} \left( |D^{q,p,\alpha} \phi_k|^2 + |D^{q,p,\alpha} \phi_{k+1}|^2 \right) d\rho d\mu \\ &+ \int_{\mathcal{B}_t} \left( ((1 + \tau)n_\tau - \rho n_\rho) |D^{q,p,\alpha} \phi_k|^2 + ((1 - \tau)n_\tau + \rho n_\rho) |D^{q,p,\alpha} \phi_{k+1}|^2 \right) d\tau d\mu \\ &- \int_{\mathcal{I}_t} \rho (|D^{q,p,\alpha} \phi_{k+1}|^2 - |D^{q,p,\alpha} \phi_k|^2) d\tau d\mu \end{aligned}$$

one obtains

$$\begin{aligned} & \int_{\mathcal{S}_t} \left( (1 + t) |D^{q,p,\alpha} \phi_k|^2 + (1 - t) |D^{q,p,\alpha} \phi_{k+1}|^2 \right) d\rho d\mu \\ &+ \int_{\mathcal{N}_t} (D^{q,p,\alpha} \bar{\phi}_k D^{q,p,\alpha} X_+ \phi_{k+1} + D^{q,p,\alpha} \phi_{k+1} D^{q,p,\alpha} X_+ \bar{\phi}_k) d\tau d\rho d\mu \\ &+ \int_{\mathcal{N}_t} (D^{q,p,\alpha} \phi_k D^{q,p,\alpha} X_- \bar{\phi}_{k+1} + D^{q,p,\alpha} \bar{\phi}_{k+1} D^{q,p,\alpha} X_- \phi_k) d\tau d\rho d\mu \\ &+ 2(p - q - k) \int_{\mathcal{N}_t} |D^{q,p,\alpha} \phi_{k+1}|^2 d\tau d\rho d\mu \\ &\leq \int_{\mathcal{S}_0} (|D^{q,p,\alpha} \phi_k|^2 + |D^{q,p,\alpha} \phi_{k+1}|^2) d\rho d\mu + 2(p - q + k - 1) \int_{\mathcal{N}_t} |D^{q,p,\alpha} \phi_k|^2 d\tau d\rho d\mu. \end{aligned} \quad (23)$$

To obtain this last inequality it has been used that

$$\int_{\mathcal{I}_t} \rho (|D^{q,p,\alpha} \phi_{k+1}|^2 - |D^{q,p,\alpha} \phi_k|^2) d\tau d\mu = 0,$$

and that

$$\int_{\mathcal{B}_t} \left( ((1 + \tau)n_\tau - \rho n_\rho) |D^{q,p,\alpha} \phi_k|^2 + ((1 - \tau)n_\tau + \rho n_\rho) |D^{q,p,\alpha} \phi_{k+1}|^2 \right) d\tau d\mu \geq 0,$$

with  $n_\tau = \nu\rho$  and  $n_\rho = \nu(1 + \tau)$  for a suitable normalisation factor  $\nu$  —recall that on the characteristic  $\mathcal{B}_t$  one has  $\rho = \rho_*/(1 + \tau)$ .

Now, consider the shifts  $q \rightarrow q'$  and  $p \rightarrow p + p'$  so that  $D^{q,p,\alpha} \phi_k \rightarrow D^{q',p+p',\alpha}(\phi_k) = D^{q',p'+\alpha}(\partial_\rho^p \phi_k)$  and  $D^{q,p,\alpha} \phi_{k+1} \rightarrow D^{q',p+p',\alpha}(\phi_{k+1}) = D^{q',p'+\alpha}(\partial_\rho^p \phi_{k+1})$ . Using lemma 1 in ap-

pendix B one has that for  $m \in \mathbb{N} \cup \{0\}$ ,

$$\begin{aligned}
& \sum_{q'+p'+|\alpha| \leq m} \int_{\mathcal{N}_t} \left( D^{q',p',\alpha}(\partial_\rho^p \bar{\phi}_k) D^{q',p',\alpha} X_+(\partial_\rho^p \phi_{k+1}) + D^{q',p',\alpha}(\partial_\rho^p \phi_{k+1}) D^{q',p',\alpha} X_+(\partial_\rho^p \bar{\phi}_k) \right) d\tau d\rho d\mu \\
&= \sum_{q'+p'+|\alpha| \leq m} \int_{\mathcal{N}_t} \left( Z^\alpha(\partial_\tau^{q'} \partial_\rho^{p'+p} \bar{\phi}_k) Z^\alpha(X_+ \partial_\tau^{q'} \partial_\rho^{p'+p} \phi_{k+1}) + Z^\alpha(\partial_\tau^{q'} \partial_\rho^{p'+p} \phi_{k+1}) Z^\alpha(X_+ \partial_\tau^{q'} \partial_\rho^{p'+p} \bar{\phi}_k) \right) d\tau d\rho d\mu \\
&= 0,
\end{aligned}$$

where  $|\alpha| = \alpha_1 + \alpha_2 + \alpha_3$ . Similarly, one has that

$$\sum_{q'+p'+|\alpha| \leq m} \int_{\mathcal{N}_t} \left( D^{q',p',\alpha}(\partial_\rho^p \phi_k) D^{q',p',\alpha} X_-(\partial_\rho^p \bar{\phi}_{k+1}) + D^{q',p',\alpha}(\partial_\rho^p \bar{\phi}_{k+1}) D^{q',p',\alpha} X_-(\partial_\rho^p \phi_k) \right) d\tau d\rho d\mu = 0.$$

Hence summing the shifted version of inequality (23) over  $q' + p' + \alpha \leq m$ , with  $m$  a suitable positive integer, one obtains

$$\begin{aligned}
& \sum_{q'+p'+|\alpha| \leq m} \int_{S_t} \left( (1+t)|D^{q',p',\alpha}(\partial_\rho^p \phi_k)|^2 + (1-t)|D^{q',p',\alpha}(\partial_\rho^p \phi_{k+1})|^2 \right) d\rho d\mu \\
&+ 2 \sum_{q'+p'+|\alpha| \leq m} (p+p'-q-k) \int_{\mathcal{N}_t} |D^{q',p',\alpha}(\partial_\rho^p \phi_{k+1})|^2 d\tau d\rho d\mu \\
&\leq \sum_{q'+p'+|\alpha| \leq m} \int_{S_0} \left( |D^{q',p',\alpha}(\partial_\rho^p \phi_k)|^2 + |D^{q',p',\alpha}(\partial_\rho^p \phi_{k+1})|^2 \right) d\rho d\mu \\
&+ 2 \sum_{q'+p'+|\alpha| \leq m} (p+p'-q'+k-1) \int_{\mathcal{N}_t} |D^{q',p',\alpha}(\partial_\rho^p \phi_k)|^2 d\rho d\mu.
\end{aligned}$$

Now, we note that for  $k = 0, 1$  on the one hand one has

$$\sum_{q'+p'+|\alpha| \leq m} (p+p'-q-k) \int_{\mathcal{N}_t} |D^{q',p',\alpha}(\partial_\rho^p \phi_{k+1})|^2 d\tau d\rho d\mu \geq (p-m-1) \sum_{q'+p'+\alpha \leq m} \int_{\mathcal{N}_t} |D^{q',p',\alpha}(\partial_\rho^p \phi_{k+1})|^2 d\tau d\rho d\mu,$$

and on the other

$$\sum_{q'+p'+|\alpha| \leq m} (p+p'-q'+k-1) \int_{\mathcal{N}_t} |D^{q',p',\alpha}(\partial_\rho^p \phi_k)|^2 d\tau d\rho d\mu \leq (p+m) \sum_{q'+p'+\alpha \leq m} \int_{\mathcal{N}_t} |D^{q',p',\alpha}(\partial_\rho^p \phi_k)|^2 d\tau d\rho d\mu.$$

Hence one obtains the inequality

$$\begin{aligned}
& \sum_{q'+p'+|\alpha| \leq m} \int_{S_t} \left( (1+t)|D^{q',p',\alpha}(\partial_\rho^p \phi_k)|^2 + (1-t)|D^{q',p',\alpha}(\partial_\rho^p \phi_{k+1})|^2 \right) d\rho d\mu \\
&+ 2(p-m-1) \sum_{q'+p'+|\alpha| \leq m} \int_{\mathcal{N}_t} |D^{q',p',\alpha}(\partial_\rho^p \phi_{k+1})|^2 d\tau d\rho d\mu \\
&\leq \sum_{q'+p'+|\alpha| \leq m} \int_{S_0} \left( |D^{q',p',\alpha}(\partial_\rho^p \phi_k)|^2 + |D^{q',p',\alpha}(\partial_\rho^p \phi_{k+1})|^2 \right) d\rho d\mu \\
&+ 2(p+m) \sum_{q'+p'+|\alpha| \leq m} \int_{\mathcal{N}_t} |D^{q',p',\alpha}(\partial_\rho^p \phi_k)|^2 d\tau d\rho d\mu. \quad (24)
\end{aligned}$$

The second term on the left hand side of the last inequality can be disregarded if  $p > m+1$  —the choice  $p = m+1$  is not useful as in the sequel one will need to estimate this terms as well. Noting that for  $t \in [0, 1]$

$$\begin{aligned}
& \sum_{q'+p'+|\alpha| \leq m} \int_{S_t} |D^{q',p',\alpha}(\partial_\rho^p \phi_k)|^2 d\rho d\mu \\
&\leq \sum_{q'+p'+|\alpha| \leq m} \int_{S_t} \left( (1+t)|D^{q',p',\alpha}(\partial_\rho^p \phi_k)|^2 + (1-t)|D^{q',p',\alpha}(\partial_\rho^p \phi_{k+1})|^2 \right) d\rho d\mu,
\end{aligned}$$

one deduces

$$\begin{aligned}
& \sum_{q'+p'+|\alpha|\leq m} \int_{\mathcal{S}_t} |D^{q',p',\alpha}(\partial_\rho^p \phi_k)|^2 d\rho d\mu \\
& \leq \sum_{q'+p'+|\alpha|\leq m} \int_{\mathcal{S}_0} (|D^{q',p',\alpha}(\partial_\rho^p \phi_k)|^2 + |D^{q',p',\alpha}(\partial_\rho^p \phi_{k+1})|^2) d\rho d\mu \\
& \quad + 2(p+m) \sum_{q'+p'+|\alpha|\leq m} \int_{\mathcal{N}_t} |D^{q',p',\alpha}(\partial_\rho^p \phi_k)|^2 d\tau d\rho d\mu, \quad (25)
\end{aligned}$$

for  $k = 0, 1$ . For later use it is noted that because the first term in inequality (24) is manifestly positive for  $t \in [0, 1]$ , then if one sets  $k = 1$  one obtains

$$\begin{aligned}
& 2(p-m-1) \sum_{q'+p'+|\alpha|\leq m} \int_{\mathcal{N}_t} |D^{q',p',\alpha}(\partial_\rho^p \phi_2)|^2 d\tau d\rho d\mu \\
& \leq \sum_{q'+p'+|\alpha|\leq m} \int_{\mathcal{S}_0} (|D^{q',p',\alpha}(\partial_\rho^p \phi_1)|^2 + |D^{q',p',\alpha}(\partial_\rho^p \phi_2)|^2) d\rho d\mu \\
& \quad + 2(p+m) \sum_{q'+p'+|\alpha|\leq m} \int_{\mathcal{N}_t} |D^{q',p',\alpha}(\partial_\rho^p \phi_1)|^2 d\tau d\rho d\mu. \quad (26)
\end{aligned}$$

We now proceed to apply Gronwall's argument to inequality (25). One can rewrite the inequality in the form

$$\frac{dy(t)}{dt} \leq f_0 + y(t), \quad (27)$$

where

$$\begin{aligned}
y(t) &= \int_0^t \left( \int_{\mathcal{S}_t} \sum_{q'+p'+|\alpha|\leq m} |D^{q',p',\alpha}(\partial_\rho^p \phi_k)|^2 d\rho d\mu \right) d\tau, \\
f_0 &= \sum_{q'+p'+|\alpha|\leq m} \int_{\mathcal{S}_0} (|D^{q',p',\alpha}(\partial_\rho^p \phi_1)|^2 + |D^{q',p',\alpha}(\partial_\rho^p \phi_2)|^2) d\rho d\mu.
\end{aligned}$$

The integration factor for (27) is  $e^{-2(p+m)t}$ , so that one obtains

$$\frac{d}{dt} \left( e^{-2(p+m)t} y(t) \right) \leq f_0 e^{-2(p+m)t}.$$

Hence, after integration, and noting that  $y(0) = 0$  one obtains

$$y(t) \leq \frac{(e^{2(p+m)t} - 1)}{2(p+m)} f_0,$$

or

$$\begin{aligned}
& \sum_{q'+p'+|\alpha|\leq m} \int_{\mathcal{N}_t} |D^{q',p',\alpha}(\partial_\rho^p \phi_k)|^2 d\tau d\rho d\mu \\
& \leq \frac{(e^{2(p+m)t} - 1)}{2(p+m)} \sum_{q'+p'+|\alpha|\leq m} \int_{\mathcal{S}_0} (|D^{q',p',\alpha}(\partial_\rho^p \phi_k)|^2 + |D^{q',p',\alpha}(\partial_\rho^p \phi_{k+1})|^2) d\rho d\mu, \quad (29)
\end{aligned}$$

for  $k = 0, 1$ . An estimate for  $D^{q',p',\alpha}(\partial_\rho^p \phi_2)$  can now, in turn, be obtained from inequalities (26) and (29). Accordingly, one obtains

$$\begin{aligned}
& \sum_{q'+p'+|\alpha|\leq m} \int_{\mathcal{N}_t} |D^{q',p',\alpha}(\partial_\rho^p \phi_2)|^2 d\tau d\rho d\mu \\
& \leq \frac{e^{2(p+m)t}}{2(p-m-1)} \sum_{q'+p'+|\alpha|\leq m} \int_{\mathcal{S}_0} (|D^{q',p',\alpha}(\partial_\rho^p \phi_1)|^2 + |D^{q',p',\alpha}(\partial_\rho^p \phi_2)|^2) d\rho d\mu.
\end{aligned}$$

Thus, for  $k = 0, 1, 2$  and  $p > m + 1$  one has the estimate

$$\begin{aligned} \int_{\mathcal{N}_t} \left( \sum_{q'+p'+|\alpha| \leq m} |D^{q',p',\alpha}(\partial_\rho^p \phi_k)|^2 \right) d\tau d\rho d\mu \\ \leq C \sum_{k=0}^2 \int_{\mathcal{S}_0} \left( \sum_{q'+p'+|\alpha| \leq m} |D^{q',p',\alpha}(\partial_\rho^p \phi_k)|^2 \right) d\rho d\mu, \end{aligned} \quad (30)$$

with  $C$  a constant depending on  $p$  and  $m$  which can be chosen independently of  $t \in [0, 1]$ .

## Discussion

As discussed thoroughly in [4], using the Sobolev embedding theorems one has that given  $t \in [0, 1]$  and  $j = 0, 1, \dots$  there is a continuous embedding  $H^{j+3}(\overset{\circ}{\mathcal{N}}_t) \rightarrow C^{j,\lambda}(\mathcal{N}_t)$ , with  $0 < \lambda < 1$ , where  $H^{j+3}$  denotes the standard  $L^2$ -type Sobolev space, and  $\overset{\circ}{\mathcal{N}}_t$  denotes the interior of the compact set  $\mathcal{N}_t$ . The space  $C^{j,\lambda}(\mathcal{N}_t)$  consists of functions in  $C^j(\overset{\circ}{\mathcal{N}}_t)$  which together with their derivatives up to order  $j$  are Hölder continuous in  $\overset{\circ}{\mathcal{N}}_t$  and thus, together with the derivatives, extend to continuous functions on  $\mathcal{N}_t$ .

For a  $\phi_k$  satisfying the estimate (30) one has then that  $\partial_\rho^p \phi_k \in C^{j,\lambda}(\mathcal{N}_t)$ ,  $t \in [0, 1]$  if  $p \geq j + 5$ . Consequently, the remainder  $R_p(\phi_k) = J^p(\partial_\rho^p \phi_k)$  in the expansion (18) is such that  $J^p(\partial_\rho^p \phi_k) \in C^{j,\lambda}(\mathcal{N}_t)$ ,  $t \in [0, 1]$ . Hence, one can prescribe the regularity of the remainder in (18) by considering an expansion with a suitably large order. On the other hand, the smoothness of the coefficients  $\phi_k^{(l)}$ ,  $0 \leq l \leq p - 1$  is known explicitly.

Finally, it is noted that if a conformal gauge for which  $\kappa \neq \rho$  in the conformal factor (2) is used, then it turns out that the argument for the construction of estimates becomes more complicated as the coefficients  $\partial_\rho^{p'} \phi_k$ ,  $p' < p$  will start appearing in the discussion. In this case, estimates can be constructed by means of a variant of the construction to be discussed in the next sections.

## 5 Estimates on Schwarzschild spacetime

In order to discuss the construction of estimates for equations (15a)-(15d) for a non-flat background one has to consider the analogues in the Schwarzschild spacetime of the sets (22a)-(22d). Given  $\rho_* > 0$ , let  $(\tilde{\tau}(s), \check{\rho}(s))$  be the solutions of the system

$$\begin{aligned} \frac{d\tilde{\tau}(s)}{ds} &= \left( 1 + \tilde{\tau}(s) + a(\tilde{\tau}(s), \check{\rho}(s)) \right), \quad \tilde{\tau}(0) = 0, \\ \frac{d\check{\rho}(s)}{ds} &= - \left( \check{\rho}(s) + b(\tilde{\tau}(s), \check{\rho}(s)) \right), \quad \check{\rho}(0) = \rho_*. \end{aligned}$$

It can be shown that there is a  $s_{\mathcal{J}^+} \in \mathbb{R}^+$  such that  $\tau(s_{\mathcal{J}^+}) = 1$  and  $\rho(s_{\mathcal{J}^+}) \neq 0$  —that is, the curve intersects future null infinity. More generally, given  $t \in [0, 1]$  there is  $s_t \in \mathbb{R}^+$  such that  $\tilde{\tau}(s_t) = t$ . Define

$$\begin{aligned} \mathcal{N}_t &= \left\{ (\tau, \rho, s) \in \mathcal{M}_{a,\omega} \mid 0 \leq \tau \leq t, 0 \leq \rho \leq \check{\rho}(s), s \in [0, s_t] \right\}, \\ \mathcal{B}_t &= \left\{ (\tau, \rho, s) \in \mathcal{M}_{a,\omega} \mid \tau = \tilde{\tau}(s), \rho = \check{\rho}(s), s \in [0, s_t] \right\}, \\ \mathcal{S}_t &= \left\{ (\tau, \rho, s) \in \mathcal{M}_{a,\omega} \mid \tau = t, 0 \leq \rho \leq \check{\rho}(s), s \in [0, s_t] \right\}, \\ \mathcal{I}_t &= \left\{ (\tau, \rho, s) \in \mathcal{M}_{a,\omega} \mid 0 \leq \tau \leq t, \rho = 0 \right\}, \end{aligned}$$

for  $t \in [0, 1]$ . Again  $\mathcal{N}_t$  is the domain of influence of  $\mathcal{S}_0 \subset \mathcal{C}_{a_*}$ .

Given  $(p, q, \alpha)$  such that  $q + p + |\alpha| = m$  for a given  $m \in \mathbb{N} \cup \{0\}$  a lengthy but straightforward calculation gives

$$\begin{aligned}
D^{q,p,\alpha} A_k &= (1 + \tau + a) \partial_\tau D^{q,p,\alpha} \phi_k - (\rho + b) \partial_\rho D^{q,p,\alpha} \phi_k + (1 + c) D^{q,p,\alpha} X_+ \phi_{k+1} \\
&\quad + ((1 - k) + f_k - p + q - p D^{0,1,0} b + q D^{1,0,0} a) D^{q,p,\alpha} \phi_k \\
&\quad + \left( \sum_{s=2}^q \binom{q}{s} D^{s,0,0} a D^{q-s+1,0,0} \phi_k + \sum_{s=1}^q \sum_{l=1}^p \binom{q}{s} \binom{p}{l} D^{s,l,0} a D^{q-s,p-l,\alpha} \phi_k \right) \\
&\quad - \left( \sum_{s=1}^q \binom{q}{s} D^{s,0,0} b D^{q-s,1,\alpha} \phi_k + \sum_{s=0}^q \sum_{l=2}^p \binom{q}{s} \binom{p}{l} D^{s,l,0} b D^{q-s,p-l+1,\alpha} \phi_k \right) \\
&\quad + \left( \sum_{s=1}^q \binom{q}{s} D^{s,0,0} c D^{q-s,p,\alpha} X_+ \phi_{k+1} + \sum_{s=0}^q \sum_{l=1}^p \binom{q}{s} \binom{p}{l} D^{s,l,0} c D^{q-s,p-l,\alpha} X_+ \phi_{k+1} \right) \\
&\quad + \left( \sum_{s=2}^q \binom{q}{s} D^{s,0,0} f_k D^{q-s,0,\alpha} \phi_k - p \sum_{s=1}^q \binom{q}{s} D^{s,1,0} f_k D^{q-s,0,\alpha} \phi_k \right) \\
&\quad + \sum_{s=0}^q \sum_{l=1}^p \binom{q}{s} \binom{p}{l} D^{s,l,0} f_k D^{q-s,p-l,\alpha} \phi_k = 0, \tag{32}
\end{aligned}$$

and

$$\begin{aligned}
D^{q,p,\alpha} B_k &= (1 - \tau - a) \partial_\tau D^{q,p,\alpha} \phi_{k+1} + (\rho + b) \partial_\rho D^{q,p,\alpha} \phi_{k+1} + (1 + c) D^{q,p,\alpha} X_- \phi_k \\
&\quad + (-k + g_k + p - q + p D^{0,1,0} b - q D^{1,0,0} a) D^{q,p,\alpha} \phi_{k+1} \\
&\quad - \left( \sum_{s=2}^q \binom{q}{s} D^{s,0,0} a D^{q-s+1,0,0} \phi_{k+1} + \sum_{s=1}^q \sum_{l=1}^p \binom{q}{s} \binom{p}{l} D^{s,l,0} a D^{q-s,p-l,\alpha} \phi_{k+1} \right) \\
&\quad + \left( \sum_{s=1}^q \binom{q}{s} D^{s,0,0} b D^{q-s,1,\alpha} \phi_{k+1} + \sum_{s=0}^q \sum_{l=2}^p \binom{q}{s} \binom{p}{l} D^{s,l,0} b D^{q-s,p-l+1,\alpha} \phi_{k+1} \right) \\
&\quad + \left( \sum_{s=1}^q \binom{q}{s} D^{s,0,0} c D^{q-s,p,\alpha} X_- \phi_{k+1} + \sum_{s=0}^q \sum_{l=1}^p \binom{q}{s} \binom{p}{l} D^{s,l,0} c D^{q-s,p-l,\alpha} X_- \phi_{k+1} \right) \\
&\quad + \left( \sum_{s=2}^q \binom{q}{s} D^{s,0,0} g_k D^{q-s,0,\alpha} \phi_{k+1} + p \sum_{s=1}^q \binom{q}{s} D^{s,1,0} g_k D^{q-s,0,\alpha} \phi_{k+1} \right) \\
&\quad + \sum_{s=0}^q \sum_{l=1}^p \binom{q}{s} \binom{p}{l} D^{s,l,0} g_k D^{q-s,p-l,\alpha} \phi_{k+1} = 0, \tag{33}
\end{aligned}$$

for  $k = 0, 1$ . It is important to note in these expressions the presence of terms of the form  $D^{q,p,\alpha} \phi_k$ ,  $k = 0, 1, 2$  with  $q + p + |\alpha| \leq m$ . These impede the straight-forward application of the methods discussed in section 4. However, it turns out that it is possible to construct estimates like those in 4 by means of an induction argument.

For later reference it will be convenient to write

$$H_a^{(q,j,\alpha)} = \sum_{s=2}^q \binom{q}{s} D^{s,0,0} a D^{q-s+1,0,\alpha} \phi_k + \sum_{s=1}^q \sum_{l=1}^{p+j} \binom{q}{s} \binom{p+j}{l} D^{s,l,0} a D^{q-s,p+j-l,\alpha} \phi_k, \quad (34a)$$

$$H_b^{(q,j,\alpha)} = - \sum_{s=1}^q \binom{q}{s} D^{s,0,0} b D^{q-s,1,\alpha} \phi_k - \sum_{s=0}^q \sum_{l=2}^{p+j} \binom{q}{s} \binom{p+j}{l} D^{s,l,0} b D^{q-s,p+j-l+1,\alpha} \phi_k, \quad (34b)$$

$$H_c^{(q,j,\alpha)} = \sum_{s=1}^q \binom{q}{s} D^{s,0,0} c D^{q-s,p,\alpha} X_+ \phi_{k+1} + \sum_{s=0}^q \sum_{l=1}^{p+j} \binom{q}{s} \binom{p+j}{l} D^{s,l,0} c D^{q-s,p+j-l,\alpha} X_- \phi_k, \quad (34c)$$

$$\begin{aligned} H_{f_k}^{(q,j,\alpha)} &= \sum_{s=2}^q \binom{q}{s} D^{s,0,0} f_k D^{q-s,0,\alpha} \phi_k - p \sum_{s=1}^q \binom{q}{s} D^{s,1,0} f_k D^{q-s,0,\alpha} \phi_k \\ &\quad + \sum_{s=0}^q \sum_{l=1}^{p+j} \binom{q}{s} \binom{p+j}{l} D^{s,l,0} f_k D^{q-s,p+j-l,\alpha} \phi_k, \end{aligned} \quad (34d)$$

$$K_a^{(q,j,\alpha)} = - \sum_{s=2}^q \binom{q}{s} D^{s,0,0} a D^{q-s+1,0,\alpha} \phi_{k+1} - \sum_{s=1}^q \sum_{l=1}^{p+j} \binom{q}{s} \binom{p+j}{l} D^{s,l,0} a D^{q-s,p+j-l,\alpha} \phi_{k+1}, \quad (34e)$$

$$K_b^{(q,j,\alpha)} = \sum_{s=1}^q \binom{q}{s} D^{s,0,0} b D^{q-s,1,\alpha} \phi_{k+1} + \sum_{s=0}^q \sum_{l=2}^{p+j} \binom{q}{s} \binom{p+j}{l} D^{s,l,0} b D^{q-s,p+j-l+1,\alpha} \phi_{k+1}, \quad (34f)$$

$$K_c^{(q,j,\alpha)} = \sum_{s=1}^q \binom{q}{s} D^{s,0,0} c D^{q-s,p,\alpha} X_+ \phi_{k+1} + \sum_{s=0}^q \sum_{l=1}^{p+j} \binom{q}{s} \binom{p+j}{l} D^{s,l,0} c D^{q-s,p+j-l,\alpha} X_+ \phi_{k+1}, \quad (34g)$$

$$\begin{aligned} K_{g_k}^{(q,j,\alpha)} &= \sum_{s=2}^q \binom{q}{s} D^{s,0,0} g_k D^{q-s,0,\alpha} \phi_{k+1} + p \sum_{s=1}^q \binom{q}{s} D^{s,1,0} g_k D^{q-s,0,\alpha} \phi_{k+1} \\ &\quad + \sum_{s=0}^q \sum_{l=1}^{p+j} \binom{q}{s} \binom{p+j}{l} D^{s,l,0} g_k D^{q-s,p+j-l,\alpha} \phi_{k+1}. \end{aligned} \quad (34h)$$

### 5.1 Estimates for $D^{q,0,\alpha}(\partial_\rho^p \phi_k)$ and $D^{q,0,\alpha}(\partial_\rho^p \phi_{k+1})$ with $q + |\alpha| \leq m$

First, it is shown how the arguments of section 4 can be adapted to obtain estimates of  $D^{q,0,\alpha}(\partial_\rho^p \phi_k)$  and  $D^{q,0,\alpha}(\partial_\rho^p \phi_{k+1})$  for  $q + |\alpha| \leq m$  and  $p$  suitably large —the latter to be determined during the argument. Again, as it is customary in the construction of estimates it is assumed that the relevant objects are as smooth as required. *In addition, it will be assumed that the components  $\phi_k$  are of the form (18).*

As in section 4, the starting point is

$$D^{q,p,\alpha} \bar{\phi}_k D^{q,p,\alpha} A_k + D^{q,p,\alpha} \phi_k D^{q,p,\alpha} \bar{A}_k + D^{q,p,\alpha} \bar{\phi}_{k+1} D^{q,p,\alpha} B_k + D^{q,p,\alpha} \phi_{k+1} D^{q,p,\alpha} \bar{B}_k = 0,$$

for  $k = 0, 1$  and  $q + |\alpha| \leq m$ . A straightforward calculation using the expressions (32) and (33)

shows that the above can be rewritten as

$$\begin{aligned}
0 = & \partial_\tau \left( (1 + \tau + a) |D^{q,0,\alpha}(\partial_\rho^p \phi_k)|^2 + (1 - \tau - a) |D^{q,0,\alpha}(\partial_\rho^p \phi_{k+1})|^2 \right) \\
& + \partial_\rho \left( (\rho + b) |D^{q,0,\alpha}(\partial_\rho^p \phi_{k+1})|^2 - (\rho + b) |D^{q,0,\alpha}(\partial_\rho^p \phi_k)|^2 \right) \\
& + (1 + c) \left( D^{q,0,\alpha}(\partial_\rho^p \bar{\phi}_k) D^{q,0,\alpha} X_+ (\partial_\rho^p \phi_{k+1}) + D^{q,0,\alpha}(\partial_\rho^p \phi_{k+1}) D^{q,0,\alpha} X_+ (\partial_\rho^p \bar{\phi}_k) \right) \\
& + (1 + c) \left( D^{q,0,\alpha}(\partial_\rho^p \phi_k) D^{q,p,\alpha} X_- (\partial_\rho^p \bar{\phi}_{k+1}) + D^{q,0,\alpha}(\partial_\rho^p \bar{\phi}_{k+1}) D^{q,0,\alpha} X_- (\partial_\rho^p \phi_k) \right) \\
& - 2(p - q + k - 1 - f_k + p D^{0,p,0} b - q D^{1,0,0} a) |D^{q,0,\alpha}(\partial_\rho^p \phi_k)|^2 \\
& + 2(p - q - k + g_k + p D^{0,1,0} b - q D^{1,0,0} a) |D^{q,0,\alpha}(\partial_\rho^p \phi_{k+1})|^2. \\
& + D^{q,0,\alpha}(\partial_\rho^p \bar{\phi}_k) H_a^{(q,0,\alpha)} + D^{q,0,\alpha}(\partial_\rho^p \phi_k) \bar{H}_a^{(q,0,\alpha)} + D^{q,0,\alpha}(\partial_\rho^p \bar{\phi}_k) H_b^{(q,0,\alpha)} + D^{q,0,\alpha}(\partial_\rho^p \phi_k) \bar{H}_b^{(q,0,\alpha)} \\
& + D^{q,0,\alpha}(\partial_\rho^p \bar{\phi}_k) H_c^{(q,0,\alpha)} + D^{q,0,\alpha}(\partial_\rho^p \phi_k) \bar{H}_c^{(q,0,\alpha)} + D^{q,0,\alpha}(\partial_\rho^p \bar{\phi}_k) H_{f_k}^{(q,0,\alpha)} + D^{q,0,\alpha}(\partial_\rho^p \phi_k) \bar{H}_{f_k}^{(q,0,\alpha)} \\
& + D^{q,0,\alpha}(\partial_\rho^p \bar{\phi}_k) K_a^{(q,0,\alpha)} + D^{q,0,\alpha}(\partial_\rho^p \phi_k) \bar{K}_a^{(q,0,\alpha)} + D^{q,0,\alpha}(\partial_\rho^p \bar{\phi}_k) K_b^{(q,0,\alpha)} + D^{q,0,\alpha}(\partial_\rho^p \phi_k) \bar{K}_b^{(q,0,\alpha)} \\
& + D^{q,0,\alpha}(\partial_\rho^p \bar{\phi}_k) K_c^{(q,0,\alpha)} + D^{q,0,\alpha}(\partial_\rho^p \phi_k) \bar{K}_c^{(q,0,\alpha)} + D^{q,0,\alpha}(\partial_\rho^p \bar{\phi}_k) K_{f_k}^{(q,0,\alpha)} + D^{q,0,\alpha}(\partial_\rho^p \phi_k) \bar{K}_{f_k}^{(q,0,\alpha)}.
\end{aligned} \tag{35}$$

Crucial in this last expression is that the highest  $\rho$ -derivatives of  $\phi_k$  and  $\phi_{k+1}$  in  $H_a^{(q,0,\alpha)}$ ,  $H_b^{(q,0,\alpha)}$ ,  $H_c^{(q,0,\alpha)}$ ,  $H_{f_k}^{(q,0,\alpha)}$ ,  $K_a^{(q,0,\alpha)}$ ,  $K_b^{(q,0,\alpha)}$ ,  $K_c^{(q,0,\alpha)}$  and  $K_{g_k}^{(q,0,\alpha)}$  —as given by the expressions (34a)-(34h)— are of order  $p-1$ . The use of the Ansatz (18) will allow to get around the problem of not having estimates for  $\partial_\rho^{p'} \phi_k$  and  $\partial_\rho^{p'} \phi_{k+1}$  for  $0 \leq p' < p$ . Indeed, if

$$\begin{aligned}
\phi_k &= \sum_{l=0}^{p-1} \frac{1}{l!} \phi_k^{(l)} \rho^l + J^p(\partial_\rho^p \phi_k), \\
\phi_{k+1} &= \sum_{l=0}^{p-1} \frac{1}{l!} \phi_{k+1}^{(l)} \rho^l + J^p(\partial_\rho^p \phi_{k+1}),
\end{aligned}$$

then a direct calculation yields

$$\partial_\rho^s \phi_k = \sum_{l=0}^{p-s-1} \frac{1}{(l-s)!} \phi_k^{(l)} \rho^{l-s} + J^{p-s}(\partial_\rho^p \phi_k),$$

for  $0 \leq s < p$ . Thus, for example, one has that

$$H_a^{(q,0,\alpha)} = \sum_{s=2}^q \binom{q}{s} D^{s,0,0} a D^{q-s+1,0,\alpha} \phi_k + \sum_{s=1}^q \sum_{l=1}^p \binom{q}{s} \binom{p}{l} D^{s,l,0} a D^{q-s,p-l,\alpha} \phi_k$$

can be rewritten as

$$H_a^{(q,0,\alpha)} = F_a^{(q,0,\alpha)} + S_a^{(q,0,\alpha)}$$

with

$$\begin{aligned}
F_a^{(q,0,\alpha)} &= \sum_{s=2}^q \sum_{r=0}^{p-1} \frac{1}{r!} \binom{q}{s} D^{s,0,0} a D^{q-s+1,0,\alpha} \phi_k^{(r)} \rho^r \\
&\quad + \sum_{s=1}^q \sum_{l=0}^p \sum_{r=0}^{l-1} \frac{1}{(r-p+l)!} \binom{q}{s} \binom{p}{l} D^{s,l,0} a D^{q-s,0,\alpha} \phi_k^{(r)} \rho^{r-p+l}, \\
S_a^{(q,0,\alpha)} &= \sum_{s=2}^q \binom{q}{s} D^{s,0,0} a J^p(D^{q-s+1,p,\alpha} \phi_k) + \sum_{s=1}^q \sum_{l=1}^p \binom{q}{s} \binom{p}{l} D^{s,l,0} a J^l(D^{q-s,p,\alpha} \phi_k).
\end{aligned}$$

The functions  $F_a^{(q,0,\alpha)}$  can be calculated explicitly for given values of  $p$ ,  $q$  and  $\alpha$ . Moreover, they can be made as smooth as necessary by an adequate choice of the initial data along the lines of the



discussion in [15]. The terms are homogeneous in  $J^i(D^{j,p,\beta}\phi_k)$ , with  $1 \leq i \leq p$  and  $j + |\beta| \leq m$ . Accordingly, given a  $\delta > 0$  there is a  $0 < \rho_* < a_*$  such that  $|S^{q,0,\alpha}| < \delta$  on  $\mathcal{N}_t$ ,  $t \in [0, 1]$ . A similar split can be performed with the other terms in (35) so that one obtains

$$\begin{aligned} H_a^{(q,0,\alpha)} &= F_a^{(q,0,\alpha)} + S_a^{(q,0,\alpha)}, & K_a^{(q,0,\alpha)} &= G_a^{(q,0,\alpha)} + R_a^{(q,0,\alpha)}, \\ H_b^{(q,0,\alpha)} &= F_b^{(q,0,\alpha)} + S_b^{(q,0,\alpha)}, & K_b^{(q,0,\alpha)} &= G_b^{(q,0,\alpha)} + R_b^{(q,0,\alpha)}, \\ H_c^{(q,0,\alpha)} &= F_c^{(q,0,\alpha)} + S_c^{(q,0,\alpha)}, & K_c^{(q,0,\alpha)} &= G_c^{(q,0,\alpha)} + R_c^{(q,0,\alpha)}, \\ H_{f_k}^{(q,0,\alpha)} &= F_{f_k}^{(q,0,\alpha)} + S_{f_k}^{(q,0,\alpha)}, & K_{g_k}^{(q,0,\alpha)} &= G_{g_k}^{(q,0,\alpha)} + R_{g_k}^{(q,0,\alpha)}, \end{aligned}$$

where again  $F_a^{(q,0,\alpha)}$ ,  $F_b^{(q,0,\alpha)}$ ,  $F_c^{(q,0,\alpha)}$ ,  $F_{f_k}^{(q,0,\alpha)}$ ,  $G_a^{(q,0,\alpha)}$ ,  $G_b^{(q,0,\alpha)}$ ,  $G_c^{(q,0,\alpha)}$  and  $G_{g_k}^{(q,0,\alpha)}$  can be calculated explicitly in terms of the solutions to the transport equations, and consequently can be made as regular as necessary by choosing a suitable choice of initial data. On the other hand  $S_a^{(q,0,\alpha)}$ ,  $S_b^{(q,0,\alpha)}$ ,  $S_c^{(q,0,\alpha)}$ ,  $S_{f_k}^{(q,0,\alpha)}$ ,  $R_a^{(q,0,\alpha)}$ ,  $R_b^{(q,0,\alpha)}$ ,  $R_c^{(q,0,\alpha)}$  and  $R_{g_k}^{(q,0,\alpha)}$  are homogeneous functions of  $J^i(D^{j,p,\beta}\phi_k)$  and  $J^i(D^{j,p,\beta}\phi_{k+1})$  with  $i \geq 1$  and  $j + |\beta| \leq m$  and hence their supremum in  $\mathcal{N}_t$  can be made suitably small by a convenient choice of  $\rho_*$ .

Substitution into (35) gives

$$\begin{aligned} 0 &= \partial_\tau \left( (1 + \tau + a) |D^{q,0,\alpha}(\partial_\rho^p \phi_k)|^2 + (1 - \tau - a) |D^{q,0,\alpha}(\partial_\rho^p \phi_{k+1})|^2 \right) \\ &\quad + \partial_\rho \left( (\rho + b) |D^{q,0,\alpha}(\partial_\rho^p \phi_{k+1})|^2 - (\rho + b) |D^{q,0,\alpha}(\partial_\rho^p \phi_k)|^2 \right) \\ &\quad + (1 + c) \left( D^{q,0,\alpha}(\partial_\rho^p \bar{\phi}_k) D^{q,0,\alpha} X_+ (\partial_\rho^p \phi_{k+1}) + D^{q,0,\alpha}(\partial_\rho^p \phi_{k+1}) D^{q,0,\alpha} X_+ (\partial_\rho^p \bar{\phi}_k) \right) \\ &\quad + (1 + c) \left( D^{q,0,\alpha}(\partial_\rho^p \phi_k) D^{q,p,\alpha} X_- (\partial_\rho^p \bar{\phi}_{k+1}) + D^{q,0,\alpha}(\partial_\rho^p \bar{\phi}_{k+1}) D^{q,0,\alpha} X_- (\partial_\rho^p \phi_k) \right) \\ &\quad - 2(p - q + k - 1 - f_k + p D^{0,p,0} b - q D^{1,0,0} a) |D^{q,0,\alpha}(\partial_\rho^p \phi_k)|^2 \\ &\quad + 2(p - q - k + g_k + p D^{0,1,0} b - q D^{1,0,0} a) |D^{q,0,\alpha}(\partial_\rho^p \phi_{k+1})|^2 \\ &\quad + D^{q,0,\alpha}(\partial_\rho^p \bar{\phi}_k) \widehat{F}_k^{(q,0,\alpha)} + D^{q,0,\alpha}(\partial_\rho^p \phi_k) \overline{\widehat{F}}_k^{(q,0,\alpha)} + D^{q,0,\alpha}(\partial_\rho^p \bar{\phi}_k) \widehat{G}_k^{(q,0,\alpha)} + D^{q,0,\alpha}(\partial_\rho^p \phi_k) \overline{\widehat{G}}_k^{(q,0,\alpha)} \\ &\quad + D^{q,0,\alpha}(\partial_\rho^p \bar{\phi}_k) \widehat{S}_k^{(q,0,\alpha)} + D^{q,0,\alpha}(\partial_\rho^p \phi_k) \overline{\widehat{S}}_k^{(q,0,\alpha)} + D^{q,0,\alpha}(\partial_\rho^p \bar{\phi}_k) \widehat{R}_k^{(q,0,\alpha)} + D^{q,0,\alpha}(\partial_\rho^p \phi_k) \overline{\widehat{R}}_k^{(q,0,\alpha)}, \end{aligned} \tag{37}$$

with

$$\begin{aligned} \widehat{F}_k^{(q,0,\alpha)} &= F_a^{(q,0,\alpha)} + F_b^{(q,0,\alpha)} + F_c^{(q,0,\alpha)} + F_{f_k}^{(q,0,\alpha)}, \\ \widehat{G}_k^{(q,0,\alpha)} &= G_a^{(q,0,\alpha)} + G_b^{(q,0,\alpha)} + G_c^{(q,0,\alpha)} + G_{g_k}^{(q,0,\alpha)}, \\ \widehat{S}_k^{(q,0,\alpha)} &= S_a^{(q,0,\alpha)} + S_b^{(q,0,\alpha)} + S_c^{(q,0,\alpha)} + S_{f_k}^{(q,0,\alpha)}, \\ \widehat{R}_k^{(q,0,\alpha)} &= R_a^{(q,0,\alpha)} + R_b^{(q,0,\alpha)} + R_c^{(q,0,\alpha)} + R_{g_k}^{(q,0,\alpha)}. \end{aligned}$$

As in the case of the Minkowski background consider the integral of (37) over  $\mathcal{N}_t$ . Using the Gauss theorem on the first two terms of equation (37) one has that

$$\begin{aligned} &\int_{\mathcal{N}_t} \partial_\tau \left( (1 + \tau + a) |D^{q,0,\alpha}(\partial_\rho^p \phi_k)|^2 + (1 - \tau - a) |D^{q,0,\alpha}(\partial_\rho^p \phi_{k+1})|^2 \right) d\tau d\rho d\mu \\ &\quad + \int_{\mathcal{N}_t} \partial_\rho \left( (\rho + b) |D^{q,0,\alpha}(\partial_\rho^p \phi_{k+1})|^2 - (\rho + b) |D^{q,0,\alpha}(\partial_\rho^p \phi_k)|^2 \right) d\tau d\rho d\mu \\ &= \int_{\mathcal{S}_t} \left( (1 + t + a) |D^{q,0,\alpha}(\partial_\rho^p \phi_k)|^2 + (1 - t - a) |D^{q,0,\alpha}(\partial_\rho^p \phi_{k+1})|^2 \right) d\rho d\mu \\ &\quad - \int_{\mathcal{S}_0} \left( |D^{q,0,\alpha}(\partial_\rho^p \phi_k)|^2 + |D^{q,0,\alpha}(\partial_\rho^p \phi_{k+1})|^2 \right) d\rho d\mu \\ &\quad + \int_{\mathcal{B}_t} \left( (1 + \tau + a) n_\tau - (\rho + b) n_\rho \right) |D^{q,0,\alpha} \phi_k|^2 \\ &\quad \quad + \left( (1 - \tau - a) n_\tau + (\rho + b) n_\rho \right) |D^{q,0,\alpha} \phi_{k+1}|^2 dnd\mu \\ &\quad - \int_{\mathcal{I}_t} (\rho + b) \left( |D^{q,0,\alpha} \phi_{k+1}|^2 - |D^{q,0,\alpha} \phi_k|^2 \right) d\tau d\mu, \end{aligned}$$

with  $n_\tau = \nu(\rho + b)$  and  $n_\rho = \nu(1 + \tau + a)$ , and  $\nu$  a normalisation factor. Note that in particular,

$$\int_{\mathcal{I}_t} (\rho + b) (|D^{q,0,\alpha} \phi_{k+1}|^2 - |D^{q,0,\alpha} \phi_k|^2) d\tau d\mu = 0,$$

and that

$$\int_{\mathcal{B}_t} ((1 + \tau + a)n_\tau - (\rho + b)n_\rho) |D^{q,0,\alpha} \phi_k|^2 + ((1 - \tau - a)n_\tau + (\rho + b)n_\rho) |D^{q,0,\alpha} \phi_{k+1}|^2 d\tau d\mu \geq 0.$$

Hence, one has that

$$\begin{aligned} & \int_{\mathcal{S}_t} \left( (1 + t + a) |D^{q,0,\alpha} (\partial_\rho^p \phi_k)|^2 + (1 - t - a) |D^{q,0,\alpha} (\partial_\rho^p \phi_{k+1})|^2 \right) d\rho d\mu \\ & + \int_{\mathcal{N}_t} (1 + c) (D^{q,0,\alpha} (\partial_\rho^p \bar{\phi}_k) D^{q,0,\alpha} X_+ (\partial_\rho^p \phi_{k+1}) + D^{q,0,\alpha} (\partial_\rho^p \phi_{k+1}) D^{q,0,\alpha} X_+ (\partial_\rho^p \bar{\phi}_k)) d\tau d\rho d\mu \\ & + \int_{\mathcal{N}_t} (1 + c) (D^{q,0,\alpha} (\partial_\rho^p \phi_k D^{q,0,\alpha} X_- (\partial_\rho^p \bar{\phi}_{k+1}) + D^{q,0,\alpha} (\partial_\rho^p \bar{\phi}_{k+1}) D^{q,0,\alpha} X_- (\partial_\rho^p \phi_k)) d\tau d\rho d\mu \\ & + 2 \int_{\mathcal{N}_t} \left( p - q - k + g_k + (p - \frac{1}{2}) D^{0,1,0} b - (q - \frac{1}{2}) D^{1,0,0} a \right) |D^{q,0,\alpha} (\partial_\rho^p \phi_{k+1})|^2 d\tau d\rho d\mu \\ & + \int_{\mathcal{N}_t} \left( D^{q,0,\alpha} (\partial_\rho^p \bar{\phi}_k) \widehat{F}_k^{(q,0,\alpha)} + D^{q,0,\alpha} (\partial_\rho^p \phi_k) \overline{\widehat{F}_k^{(q,0,\alpha)}} \right) d\tau d\rho d\mu \\ & + \int_{\mathcal{N}_t} \left( D^{q,0,\alpha} (\partial_\rho^p \bar{\phi}_k) \widehat{G}_k^{(q,0,\alpha)} + D^{q,0,\alpha} (\partial_\rho^p \phi_k) \overline{\widehat{G}_k^{(q,0,\alpha)}} \right) d\tau d\rho d\mu \\ & + \int_{\mathcal{N}_t} \left( D^{q,0,\alpha} (\partial_\rho^p \bar{\phi}_k) \widehat{S}_k^{(q,0,\alpha)} + D^{q,0,\alpha} (\partial_\rho^p \phi_k) \overline{\widehat{S}_k^{(q,0,\alpha)}} \right) d\tau d\rho d\mu \\ & + \int_{\mathcal{N}_t} \left( D^{q,0,\alpha} (\partial_\rho^p \bar{\phi}_k) \widehat{R}_k^{(q,0,\alpha)} + D^{q,0,\alpha} (\partial_\rho^p \phi_k) \overline{\widehat{R}_k^{(q,0,\alpha)}} \right) d\tau d\rho d\mu \\ & \leq \int_{\mathcal{S}_0} (|D^{q,0,\alpha} (\partial_\rho^p \phi_k)|^2 + |D^{q,0,\alpha} (\partial_\rho^p \phi_{k+1})|^2) d\rho d\mu \\ & + 2 \int_{\mathcal{N}_t} \left( p - q + k - 1 - f_k + (p - \frac{1}{2}) D^{0,1,0} b - (q - \frac{1}{2}) D^{1,0,0} a \right) |D^{q,0,\alpha} (\partial_\rho^p \phi_k)|^2 d\tau d\rho d\mu. \end{aligned} \tag{38}$$

In the sequel it shall be used that

$$\left| \int_{\mathcal{N}_t} \left( D^{q,0,\alpha} (\partial_\rho^p \bar{\phi}_k) \widehat{F}_k + D^{q,0,\alpha} (\partial_\rho^p \phi_k) \overline{\widehat{F}_k} \right) d\tau d\rho d\mu \right| \leq \int_{\mathcal{N}_t} \left( \varepsilon |D^{q,0,\alpha} (\partial_\rho^p \phi_k)|^2 + \frac{1}{\varepsilon} |\widehat{F}_k|^2 \right) d\tau d\rho d\mu, \tag{39}$$

$$\left| \int_{\mathcal{N}_t} \left( D^{q,0,\alpha} (\partial_\rho^p \bar{\phi}_k) \widehat{G}_k + D^{q,0,\alpha} (\partial_\rho^p \phi_k) \overline{\widehat{G}_k} \right) d\tau d\rho d\mu \right| \leq \int_{\mathcal{N}_t} \left( \varepsilon |D^{q,0,\alpha} (\partial_\rho^p \phi_k)|^2 + \frac{1}{\varepsilon} |\widehat{G}_k|^2 \right) d\tau d\rho d\mu, \tag{40}$$

for some small  $\varepsilon > 0$ . Particular attention will be given to controlling the last two terms in the left hand side of inequality (38). To this end choose  $\rho_*$  such that

$$|\widehat{S}_k^{(q,0,\alpha)}| \leq \delta, \quad |\widehat{R}_k^{(q,0,\alpha)}| \leq \delta,$$

for  $(\tau, \rho, \varsigma) \in \mathcal{N}_t$  and  $\delta$  a suitably small non-negative number: this can always be done as  $\widehat{S}_k^{(q,0,\alpha)}$  and  $\widehat{R}_k^{(q,0,\alpha)}$  are homogeneous functions of  $J^i(D^{j,p,\beta} \phi_k)$  and  $J^i(D^{j,p,\beta} \phi_{k+1})$  with  $i \geq 1$  and  $j +$

$|\beta| \leq m$ . Further, one has that

$$\begin{aligned}
& \left| \int_{\mathcal{N}_t} \left( D^{q,0,\alpha}(\partial_\rho^p \bar{\phi}_k) \widehat{S}_k^{(q,0,\alpha)} + D^{q,0,\alpha}(\partial_\rho^p \phi_k) \widetilde{S}_k^{(q,0,\alpha)} \right) d\tau d\rho d\mu \right| \\
& \leq 2 \int_{\mathcal{N}_t} |D^{q,0,\alpha}(\partial_\rho^p \phi_k)| \left| \widehat{S}_k^{(q,0,\alpha)} \right| d\tau d\rho d\mu \\
& \leq 2\delta \int_{\mathcal{N}_t} |D^{q,0,\alpha}(\partial_\rho^p \phi_k)| d\tau d\rho d\mu \\
& \leq 2\delta \int_{\mathcal{N}_t} (1 + |D^{q,0,\alpha}(\partial_\rho^p \phi_k)|^2) d\tau d\rho d\mu. \tag{41}
\end{aligned}$$

Similarly, one finds that

$$\left| \int_{\mathcal{N}_t} \left( D^{q,0,\alpha}(\partial_\rho^p \bar{\phi}_k) \widehat{R}_k^{(q,0,\alpha)} + D^{q,0,\alpha}(\partial_\rho^p \phi_k) \widetilde{R}_k^{(q,0,\alpha)} \right) d\tau d\rho d\mu \right| \leq 2\delta \int_{\mathcal{N}_t} (1 + |D^{q,0,\alpha}(\partial_\rho^p \phi_k)|^2) d\tau d\rho d\mu, \tag{42}$$

for suitable  $\rho_*$ . Using the inequalities (41) and (42) together with (39) and (40) one arrives at

$$\begin{aligned}
& \int_{\mathcal{S}_t} \left( (1+t+a)|D^{q,0,\alpha}(\partial_\rho^p \phi_k)|^2 + (1-t-a)|D^{q,0,\alpha}(\partial_\rho^p \phi_{k+1})|^2 \right) d\rho d\mu \\
& + \int_{\mathcal{N}_t} (1+c) \left( D^{q,0,\alpha}(\partial_\rho^p \bar{\phi}_k) D^{q,0,\alpha} X_+(\partial_\rho^p \phi_{k+1}) + D^{q,0,\alpha}(\partial_\rho^p \phi_{k+1}) D^{q,0,\alpha} X_+(\partial_\rho^p \bar{\phi}_k) \right) d\tau d\rho d\mu \\
& + \int_{\mathcal{N}_t} (1+c) \left( D^{q,0,\alpha}(\partial_\rho^p \phi_k) D^{q,0,\alpha} X_-(\partial_\rho^p \bar{\phi}_{k+1}) + D^{q,0,\alpha}(\partial_\rho^p \bar{\phi}_{k+1}) D^{q,0,\alpha} X_-(\partial_\rho^p \phi_k) \right) d\tau d\rho d\mu \\
& + 2 \int_{\mathcal{N}_t} \left( p - q - k - \frac{\varepsilon}{2} - \delta + g_k + \left(p - \frac{1}{2}\right) D^{0,1,0} b - \left(q - \frac{1}{2}\right) D^{1,0,0} a \right) |D^{q,0,\alpha}(\partial_\rho^p \phi_{k+1})|^2 d\tau d\rho d\mu \\
& \leq \int_{\mathcal{S}_0} (|D^{q,0,\alpha}(\partial_\rho^p \phi_k)|^2 + |D^{q,0,\alpha}(\partial_\rho^p \phi_{k+1})|^2) d\rho d\mu \\
& + 2 \int_{\mathcal{N}_t} \left( p - q + k - f_k - 1 + \frac{\varepsilon}{2} + \delta + \left(p - \frac{1}{2}\right) D^{0,1,0} b - \left(q - \frac{1}{2}\right) D^{1,0,0} a \right) |D^{q,0,\alpha}(\partial_\rho^p \phi_k)|^2 d\tau d\rho d\mu \\
& + \int_{\mathcal{N}_t} \left( 2\delta + \frac{1}{\varepsilon} |\widehat{F}_k^{(q,0,\alpha)}|^2 + \frac{1}{\varepsilon} |\widehat{G}_k^{(q,0,\alpha)}|^2 \right) d\tau d\rho d\mu.
\end{aligned}$$

At this point we note that

$$\begin{aligned}
f_k &= \mathcal{O}(\rho), & D^{1,0,0} a &= \mathcal{O}(\rho), \\
g_k &= \mathcal{O}(\rho), & D^{0,1,0} b &= \mathcal{O}(\rho).
\end{aligned}$$

Hence —by making  $\rho_*$  even smaller, if necessary— one can make

$$\begin{aligned}
& \left| g_k + \left(p - \frac{1}{2}\right) D^{0,1,0} b - \left(q - \frac{1}{2}\right) D^{1,0,0} a \right| \leq \eta, \\
& \left| -f_k + \left(p - \frac{1}{2}\right) D^{0,1,0} b - \left(q - \frac{1}{2}\right) D^{1,0,0} a \right| \leq \eta,
\end{aligned}$$

for suitably small  $\eta > 0$ . Consequently one arrives to the inequality

$$\begin{aligned}
& \int_{\mathcal{S}_t} \left( (1+t+a)|D^{q,0,\alpha}(\partial_\rho^p \phi_k)|^2 + (1-t-a)|D^{q,0,\alpha}(\partial_\rho^p \phi_{k+1})|^2 \right) d\rho d\mu \\
& + \int_{\mathcal{N}_t} (1+c) \left( D^{q,0,\alpha}(\partial_\rho^p \bar{\phi}_k) D^{q,0,\alpha} X_+(\partial_\rho^p \phi_{k+1}) + D^{q,0,\alpha}(\partial_\rho^p \phi_{k+1}) D^{q,0,\alpha} X_+(\partial_\rho^p \bar{\phi}_k) \right) d\tau d\rho d\mu \\
& + \int_{\mathcal{N}_t} (1+c) \left( D^{q,0,\alpha}(\partial_\rho^p \phi_k) D^{q,0,\alpha} X_-(\partial_\rho^p \bar{\phi}_{k+1}) + D^{q,0,\alpha}(\partial_\rho^p \bar{\phi}_{k+1}) D^{q,0,\alpha} X_-(\partial_\rho^p \phi_k) \right) d\tau d\rho d\mu \\
& + 2 \int_{\mathcal{N}_t} \left( p - q - k - \frac{\varepsilon}{2} - \delta - \eta \right) |D^{q,0,\alpha}(\partial_\rho^p \phi_{k+1})|^2 d\tau d\rho d\mu \\
& \leq \int_{\mathcal{S}_0} (|D^{q,0,\alpha}(\partial_\rho^p \phi_k)|^2 + |D^{q,0,\alpha}(\partial_\rho^p \phi_{k+1})|^2) d\rho d\mu \\
& + 2 \int_{\mathcal{N}_t} \left( p - q + k - 1 + \frac{\varepsilon}{2} + \delta + \eta \right) |D^{q,0,\alpha}(\partial_\rho^p \phi_k)|^2 d\tau d\rho d\mu \\
& + \int_{\mathcal{N}_t} \left( 2\delta + \frac{1}{\varepsilon} |\widehat{F}_k^{(q,0,\alpha)}|^2 + \frac{1}{\varepsilon} |\widehat{G}_k^{(q,0,\alpha)}|^2 \right) d\tau d\rho d\mu,
\end{aligned}$$

for suitably small  $\rho_*$ . The next step is to sum over  $q$  and  $\alpha$  for  $q + |\alpha| \leq m$ . One obtains

$$\begin{aligned}
& \sum_{q+|\alpha| \leq m} \int_{\mathcal{S}_t} \left( (1+t+a)|D^{q,0,\alpha}(\partial_\rho^p \phi_k)|^2 + (1-t-a)|D^{q,0,\alpha}(\partial_\rho^p \phi_{k+1})|^2 \right) d\rho d\mu \\
& + 2 \sum_{q+|\alpha| \leq m} \left( p - q - k - \frac{\varepsilon}{2} - \delta - \eta \right) \int_{\mathcal{N}_t} |D^{q,0,\alpha}(\partial_\rho^p \phi_{k+1})|^2 d\tau d\rho d\mu \\
& \leq \sum_{q+|\alpha| \leq m} \int_{\mathcal{S}_0} (|D^{q,0,\alpha}(\partial_\rho^p \phi_k)|^2 + |D^{q,0,\alpha}(\partial_\rho^p \phi_{k+1})|^2) d\rho d\mu \\
& + 2 \sum_{q+|\alpha| \leq m} \left( p - q + k - 1 + \frac{\varepsilon}{2} + \delta + \eta \right) \int_{\mathcal{N}_t} |D^{q,0,\alpha}(\partial_\rho^p \phi_k)|^2 d\tau d\rho d\mu \\
& + \sum_{q+|\alpha| \leq m} \int_{\mathcal{N}_t} \left( 2\delta + \frac{1}{\varepsilon} |\widehat{F}_k^{(q,0,\alpha)}|^2 + \frac{1}{\varepsilon} |\widehat{G}_k^{(q,0,\alpha)}|^2 \right) d\tau d\rho d\mu,
\end{aligned}$$

where it has been used that

$$\begin{aligned}
& \sum_{q+|\alpha| \leq m} \int_{\mathcal{N}_t} (1+c) \left( D^{q,0,\alpha}(\partial_\rho^p \bar{\phi}_k) D^{q,0,\alpha} X_+(\partial_\rho^p \phi_{k+1}) + D^{q,0,\alpha}(\partial_\rho^p \phi_{k+1}) D^{q,0,\alpha} X_+(\partial_\rho^p \bar{\phi}_k) \right) d\tau d\rho d\mu = 0 \\
& \sum_{q+|\alpha| \leq m} \int_{\mathcal{N}_t} (1+c) \left( D^{q,0,\alpha}(\partial_\rho^p \phi_k) D^{q,0,\alpha} X_-(\partial_\rho^p \bar{\phi}_{k+1}) + D^{q,0,\alpha}(\partial_\rho^p \bar{\phi}_{k+1}) D^{q,0,\alpha} X_-(\partial_\rho^p \phi_k) \right) d\tau d\rho d\mu = 0,
\end{aligned}$$

as a consequence of lemma 1 in appendix B and of  $Xc = 0$ ,  $X_\pm c = 0$ . Note that for  $k = 0, 1$  one has that

$$\begin{aligned}
& 2 \sum_{q+|\alpha| \leq m} \left( p - q - k - \frac{\varepsilon}{2} - \delta - \eta \right) \int_{\mathcal{N}_t} |D^{q,0,\alpha}(\partial_\rho^p \phi_{k+1})|^2 d\tau d\rho d\mu \\
& \geq 2 \left( p - m - 1 - \frac{\varepsilon}{2} - \delta - \eta \right) \sum_{q+|\alpha| \leq m} \int_{\mathcal{N}_t} |D^{q,0,\alpha}(\partial_\rho^p \phi_{k+1})|^2 d\tau d\rho d\mu, \\
& 2 \sum_{q+|\alpha| \leq m} \left( p - q + k - 1 + \frac{\varepsilon}{2} + \delta + \eta \right) \int_{\mathcal{N}_t} |D^{q,0,\alpha}(\partial_\rho^p \phi_k)|^2 d\tau d\rho d\mu \\
& \leq 2 \left( p + \frac{\varepsilon}{2} + \delta + \eta \right) \sum_{q+|\alpha| \leq m} \int_{\mathcal{N}_t} |D^{q,0,\alpha}(\partial_\rho^p \phi_k)|^2 d\tau d\rho d\mu,
\end{aligned}$$

from where it follows that

$$\begin{aligned}
& \sum_{q+|\alpha|\leq m} \int_{\mathcal{S}_t} \left( (1+t+a) |D^{q,0,\alpha}(\partial_\rho^p \phi_k)|^2 + (1-t-a) |D^{q,0,\alpha} \partial_\rho^p(\phi_{k+1})|^2 \right) d\rho d\mu \\
& + 2 \left( p-m-1 - \frac{\varepsilon}{2} - \delta - \eta \right) \sum_{q+|\alpha|\leq m} \int_{\mathcal{N}_t} |D^{q,0,\alpha}(\partial_\rho^p \phi_{k+1})|^2 d\tau d\rho d\mu \\
& \leq \sum_{q+|\alpha|\leq m} \int_{\mathcal{S}_0} (|D^{q,0,\alpha}(\partial_\rho^p \phi_k)|^2 + |D^{q,0,\alpha}(\partial_\rho^p \phi_{k+1})|^2) d\rho d\mu \\
& + 2 \left( p + \frac{\varepsilon}{2} + \delta + \eta \right) \sum_{q+|\alpha|\leq m} \int_{\mathcal{N}_t} |D^{q,0,\alpha}(\partial_\rho^p \phi_k)|^2 d\tau d\rho d\mu \\
& + \sum_{q+|\alpha|\leq m} \int_{\mathcal{N}_t} \left( 2\delta + \frac{1}{\varepsilon} |\widehat{F}_k^{(q,0,\alpha)}|^2 + \frac{1}{\varepsilon} |\widehat{G}_k^{(q,0,\alpha)}|^2 \right) d\tau d\rho d\mu. \quad (43)
\end{aligned}$$

In order to guarantee that the second term on the left hand side of inequality (43) is positive —so that it can be removed from the inequality—, choose  $p$  such that  $p > m+1 + \varepsilon/2 + \delta + \eta$ . Hence  $\rho_*$  will be suitably chosen such that  $\varepsilon/2 + \delta + \eta < 1$ . Further, for  $t \in [0, 1]$  it holds that

$$\int_{\mathcal{S}_t} |D^{q,0,\alpha}(\partial_\rho^p \phi_k)|^2 d\rho d\mu \leq \int_{\mathcal{S}_t} \left( (1+t+a) |D^{q,0,\alpha}(\partial_\rho^p \phi_k)|^2 + (1-t-a) |D^{q,0,\alpha} \partial_\rho^p(\phi_{k+1})|^2 \right) d\rho d\mu,$$

for suitably small  $\rho_* > 0$  such that  $(1+t+a) \geq 1$  on  $\mathcal{S}_t$  —note that as discussed in section 2, equation (10)  $(\tau+a)|_{\mathcal{S}^+} = 1$ — and  $(1-t-a) \geq 0$ . Hence one arrives to the inequality

$$\begin{aligned}
\sum_{q+|\alpha|\leq m} \int_{\mathcal{S}_t} |D^{q,0,\alpha}(\partial_\rho^p \phi_k)|^2 d\rho d\mu & \leq \sum_{q+|\alpha|\leq m} \int_{\mathcal{S}_0} (|D^{q,0,\alpha}(\partial_\rho^p \phi_k)|^2 + |D^{q,0,\alpha}(\partial_\rho^p \phi_{k+1})|^2) d\rho d\mu \\
& + 2(p+1) \sum_{q+|\alpha|\leq m} \int_{\mathcal{N}_t} |D^{q,0,\alpha}(\partial_\rho^p \phi_k)|^2 d\tau d\rho d\mu \\
& + \sum_{q+|\alpha|\leq m} \int_{\mathcal{N}_t} \left( 2\delta + \frac{1}{\varepsilon} |\widehat{F}_k^{(q,0,\alpha)}|^2 + \frac{1}{\varepsilon} |\widehat{G}_k^{(q,0,\alpha)}|^2 \right) d\tau d\rho d\mu, \quad (44)
\end{aligned}$$

for  $p > m+2$  for suitably small  $\rho_*$  and  $k = 0, 1$ . Note also that

$$\begin{aligned}
2(p-m-2) \sum_{q+|\alpha|\leq m} \int_{\mathcal{N}_t} |D^{q,0,\alpha}(\partial_\rho^p \phi_{k+1})|^2 d\tau d\rho d\mu \\
\leq \sum_{q+|\alpha|\leq m} \int_{\mathcal{S}_0} (|D^{q,0,\alpha}(\partial_\rho^p \phi_k)|^2 + |D^{q,0,\alpha}(\partial_\rho^p \phi_{k+1})|^2) d\rho d\mu \\
+ 2(p+1) \sum_{q+|\alpha|\leq m} \int_{\mathcal{N}_t} |D^{q,0,\alpha}(\partial_\rho^p \phi_k)|^2 d\tau d\rho d\mu \\
+ \sum_{q+|\alpha|\leq m} \int_{\mathcal{N}_t} \left( 2\delta + \frac{1}{\varepsilon} |\widehat{F}_k^{(q,0,\alpha)}|^2 + \frac{1}{\varepsilon} |\widehat{G}_k^{(q,0,\alpha)}|^2 \right) d\tau d\rho d\mu, \quad (45)
\end{aligned}$$

for  $t \in [0, 1]$  and suitably small  $\rho_*$ . Applying the standard Gronwall argument to inequality (44)

one gets

$$\begin{aligned}
& \sum_{q+|\alpha|\leq m} \int_{\mathcal{N}_t} |D^{q,0,\alpha}(\partial_\rho^p \phi_k)|^2 d\tau d\rho d\mu \\
& \leq \frac{1}{2(p+1)} \left( e^{2(p+1)t} - 1 \right) \sum_{q+|\alpha|\leq m} \int_{\mathcal{S}_0} (|D^{q,0,\alpha}(\partial_\rho^p \phi_k)|^2 + |D^{q,0,\alpha}(\partial_\rho^p \phi_{k+1})|^2) d\rho d\mu \\
& \quad + e^{2(p+1)t} \int_0^t e^{-2(p+1)s} \left( \sum_{q+|\alpha|\leq m} \int_{\mathcal{N}_t} \left( 2\delta + \frac{1}{\varepsilon} |\widehat{F}_k^{(q,0,\alpha)}|^2 + \frac{1}{\varepsilon} |\widehat{G}_k^{(q,0,\alpha)}|^2 \right) d\tau d\rho d\mu \right) ds,
\end{aligned} \tag{46}$$

for  $k = 0, 1$ ,  $t \in [0, 1]$ ,  $p > m + 2$  and suitably small  $\rho_*$ . Note that the second term in the right hand side of the last inequality is bounded for  $t \in [0, 1]$ . In order to obtain an estimate for  $D^{q,0,\alpha}(\partial_\rho^p \phi_2)$ , use inequality (45) in conjunction with (46) so that one obtains:

$$\begin{aligned}
& \sum_{q+|\alpha|\leq m} \int_{\mathcal{N}_t} |D^{q,0,\alpha}(\partial_\rho^p \phi_2)|^2 d\tau d\rho d\mu \\
& \leq \frac{e^{2(p+1)t}}{2(p-m-2)} \sum_{q+|\alpha|\leq m} \int_{\mathcal{S}_0} (|D^{q,0,\alpha}(\partial_\rho^p \phi_1)|^2 + |D^{q,0,\alpha}(\partial_\rho^p \phi_2)|^2) d\rho d\mu \\
& \quad + \frac{2(p+1)}{2(p-m-2)} e^{2(p+1)t} \int_0^t e^{-2(p+1)s} \left( \sum_{q+|\alpha|\leq m} \int_{\mathcal{N}_t} \left( 2\delta + \frac{1}{\varepsilon} |\widehat{F}_1^{(q,0,\alpha)}|^2 + \frac{1}{\varepsilon} |\widehat{G}_1^{(q,0,\alpha)}|^2 \right) d\tau d\rho d\mu \right) ds \\
& \quad + \frac{1}{2(p-m-2)} \sum_{q+|\alpha|\leq m} \int_{\mathcal{N}_t} \left( 2\delta + \frac{1}{\varepsilon} |\widehat{F}_1^{(q,0,\alpha)}|^2 + \frac{1}{\varepsilon} |\widehat{G}_1^{(q,0,\alpha)}|^2 \right) d\tau d\rho d\mu,
\end{aligned}$$

for  $p > m + 2$ . Again the last two terms on the right hand side of this last inequality are bounded at  $\tau = 1$ . Hence, given  $p > m + 2$  one can conclude that there is a  $\rho_* > 0$ , and positive constants  $C_1(\rho_*)$  and  $C_2(\rho_*)$  depending on  $\rho_*$  such that

$$\begin{aligned}
& \int_{\mathcal{N}_t} \left( \sum_{q+|\alpha|\leq m} |D^{q,0,\alpha}(\partial_\rho^p \phi_k)|^2 \right) d\tau d\rho d\mu \\
& \leq C_1(\rho_*) \sum_{k=0}^2 \int_{\mathcal{S}_0} \left( \sum_{q+|\alpha|\leq m} |D^{q,0,\alpha}(\partial_\rho^p \phi_k)|^2 \right) d\rho d\mu + C_2(\rho_*).
\end{aligned} \tag{47}$$

This inequality will be used as the base step in a finite inductive argument.

## 5.2 Estimates for $D^{q,j,\alpha}(\partial_\rho^p \phi_k)$ and $D^{q,j,\alpha}(\partial_\rho^p \phi_{k+1})$ , $q + j + |\alpha| \leq m$

One can construct estimates for  $D^{q,p',\alpha}(\partial_\rho^p \phi_k)$ ,  $D^{q,p',\alpha}(\partial_\rho^p \phi_{k+1})$ ,  $q + p' + |\alpha| \leq m$ , where  $m$  is an arbitrary, but fixed non-negative integer by means of an inductive argument on  $p'$ . The base step ( $p' = 0$ ) of this induction argument is given by the estimate (47) obtained in the previous subsection. Accordingly, it shall be assumed that there exist a suitably small  $\rho_* > 0$  and constants  $C_1(\rho_*)$  and  $C_2(\rho_*)$  depending on  $\rho_*$  for which the quantities  $D^{q,j,\alpha}(\partial_\rho^p \phi_k)$ ,  $D^{q,j,\alpha}(\partial_\rho^p \phi_{k+1})$ , with  $0 \leq j \leq p' - 1 < m$  satisfy estimates of the form

$$\begin{aligned}
& \int_{\mathcal{N}_t} \sum_{\substack{q+j+|\alpha|\leq m \\ 0 \leq j < p'}} |D^{q,j,\alpha}(\partial_\rho^p \phi_k)|^2 d\tau d\rho d\mu \\
& \leq C_1(\rho_*) \sum_{k=0}^2 \int_{\mathcal{S}_0} \left( \sum_{\substack{q+j+|\alpha|\leq m \\ 0 \leq j < p'}} |D^{q,j,\alpha}(\partial_\rho^p \phi_k)|^2 \right) d\rho d\mu + C_2(\rho_*),
\end{aligned} \tag{48}$$

for  $k = 0, 1, 2$ . It will be shown that analogous estimates hold for  $0 \leq j \leq p' \leq m$ .

Using the expressions (32) and (33) one can rewrite

$$D^{q,p+j,\alpha} \bar{\phi}_k D^{q,p+j,\alpha} A_k + D^{q,p+j,\alpha} \phi_k D^{q,p+j,\alpha} \bar{A}_k + D^{q,p+j,\alpha} \bar{\phi}_k D^{q,p+j,\alpha} B_k + D^{q,p+j,\alpha} \phi_k D^{q,p+j,\alpha} \bar{B}_k = 0,$$

for  $k = 0, 1$  and  $0 \leq j \leq m$  as

$$\begin{aligned} 0 = & \partial_\tau \left( (1 + \tau + a) |D^{q,j,\alpha}(\partial_\rho^p \phi_k)|^2 + (1 - \tau - a) |D^{q,j,\alpha}(\partial_\rho^p \phi_{k+1})|^2 \right) \\ & + \partial_\rho \left( (\rho + b) |D^{q,j,\alpha}(\partial_\rho^p \phi_{k+1})|^2 - (\rho + b) |D^{q,j,\alpha}(\partial_\rho^p \phi_k)|^2 \right) \\ & + (1 + c) \left( D^{q,j,\alpha}(\partial_\rho^p \bar{\phi}_k) D^{q,j,\alpha} X_+ (\partial_\rho^p \phi_{k+1}) + D^{q,j,\alpha}(\partial_\rho^p \phi_{k+1}) D^{q,j,\alpha} X_+ (\partial_\rho^p \bar{\phi}_k) \right) \\ & + (1 + c) \left( D^{q,j,\alpha}(\partial_\rho^p \phi_k) D^{q,j,\alpha} X_- (\partial_\rho^p \bar{\phi}_{k+1}) + D^{q,j,\alpha}(\partial_\rho^p \bar{\phi}_{k+1}) D^{q,j,\alpha} X_- (\partial_\rho^p \phi_k) \right) \\ & - 2 \left( p + j - q + k - 1 - f_k + \left( p - \frac{1}{2} \right) D^{0,1,0} b - \left( q - \frac{1}{2} \right) D^{1,0,0} a \right) |D^{q,j,\alpha}(\partial_\rho^p \phi_k)|^2 \\ & + 2 \left( p + j - q - k + g_k + \left( p - \frac{1}{2} \right) D^{0,1,0} b - \left( q - \frac{1}{2} \right) D^{1,0,0} a \right) |D^{q,j,\alpha}(\partial_\rho^p \phi_{k+1})|^2. \\ & + D^{q,j,\alpha}(\partial_\rho^p \bar{\phi}_k) H_a^{(q,j,\alpha)} + D^{q,j,\alpha}(\partial_\rho^p \phi_k) \bar{H}_a^{(q,j,\alpha)} + D^{q,j,\alpha}(\partial_\rho^p \bar{\phi}_k) H_b^{(q,j,\alpha)} + D^{q,j,\alpha}(\partial_\rho^p \phi_k) \bar{H}_b^{(q,j,\alpha)} \\ & + D^{q,j,\alpha}(\partial_\rho^p \bar{\phi}_k) H_c^{(q,j,\alpha)} + D^{q,j,\alpha}(\partial_\rho^p \phi_k) \bar{H}_c^{(q,j,\alpha)} + D^{q,j,\alpha}(\partial_\rho^p \bar{\phi}_k) H_{f_k}^{(q,j,\alpha)} + D^{q,j,\alpha}(\partial_\rho^p \phi_k) \bar{H}_{f_k}^{(q,j,\alpha)} \\ & + D^{q,j,\alpha}(\partial_\rho^p \bar{\phi}_k) K_a^{(q,j,\alpha)} + D^{q,j,\alpha}(\partial_\rho^p \phi_k) \bar{K}_a^{(q,j,\alpha)} + D^{q,j,\alpha}(\partial_\rho^p \bar{\phi}_k) K_b^{(q,j,\alpha)} + D^{q,j,\alpha}(\partial_\rho^p \phi_k) \bar{K}_b^{(q,j,\alpha)} \\ & + D^{q,j,\alpha}(\partial_\rho^p \bar{\phi}_k) K_c^{(q,j,\alpha)} + D^{q,j,\alpha}(\partial_\rho^p \phi_k) \bar{K}_c^{(q,j,\alpha)} + D^{q,j,\alpha}(\partial_\rho^p \bar{\phi}_k) K_{g_k}^{(q,j,\alpha)} + D^{q,j,\alpha}(\partial_\rho^p \phi_k) \bar{K}_{g_k}^{(q,j,\alpha)}, \end{aligned}$$

with  $H_a^{(q,j,\alpha)}$ ,  $H_b^{(q,j,\alpha)}$ ,  $H_c^{(q,j,\alpha)}$ ,  $H_{f_k}^{(q,j,\alpha)}$  and  $K_a^{(q,j,\alpha)}$ ,  $K_b^{(q,j,\alpha)}$ ,  $K_c^{(q,j,\alpha)}$ ,  $K_{g_k}^{(q,j,\alpha)}$  given by formulae (34a)-(34h). As in subsection 5.1, a detailed analysis of these terms will be crucial. For example, one has that

$$\begin{aligned} H_a^{(q,j,\alpha)} &= \sum_{s=2}^q \binom{q}{s} D^{s,0,0} a D^{q-s+1,0,\alpha} \phi_k + \sum_{s=1}^q \sum_{l=1}^{p+j} \binom{q}{s} \binom{p+j}{l} D^{s,l,0} a D^{q-s,p+j-l,\alpha} \phi_k \\ &= \sum_{s=2}^q \binom{q}{s} D^{s,0,0} a D^{q-s+1,0,\alpha} \phi_k \\ &\quad + \sum_{s=1}^q \sum_{l=1}^j \binom{q}{s} \binom{p+j}{l} D^{s,l,0} a D^{q-s,p+j-l,\alpha} \phi_k \\ &\quad + \sum_{s=1}^q \sum_{l=j+1}^{p+j} \binom{q}{s} \binom{p+j}{l} D^{s,l,0} a D^{q-s,p+j-l,\alpha} \phi_k. \end{aligned}$$

The terms in the second and fourth lines of the last formula contain at most  $\rho$ -derivatives of  $\phi_k$  of order  $p - 1$ . Thus, they can be handled as in subsection 5.1 by using the expansion Ansatz (18). Note however, that this approach is not applicable to the expression in the third line as it contains  $\rho$ -derivatives of  $\phi_k$  of order  $p$  and higher. These terms will be controlled by means of the induction hypothesis for  $0 \leq j \leq p' - 1 < m$ . Using the expansion Ansatz (18) one has then

that

$$\begin{aligned}
H_a^{(q,j,\alpha)} &= \sum_{s=2}^q \sum_{l=0}^{p-1} \frac{1}{l!} \binom{q}{s} D^{s,0,0} a D^{q-s+1,0,\alpha} \phi_k^{(l)} \rho^l \\
&\quad + \sum_{s=2}^q \binom{q}{s} D^{s,0,0} a J^p (D^{q-s+1,0,\alpha} \phi_k) \\
&\quad + U_a^{(q,j,\alpha)} \\
&\quad + \sum_{s=1}^q \sum_{l=j+1}^{p+j} \sum_{t=p+j-l}^{p-1} \frac{1}{(t-p-j+l)!} \binom{q}{s} \binom{p+j}{l} D^{s,l,\alpha} a D^{q-s,0,\alpha} \phi_k^{(t)} \rho^{t-p-j+l} \\
&\quad + \sum_{s=1}^q \sum_{l=j+1}^{p+j} \binom{q}{s} \binom{p+j}{l} D^{s,l,0} a J^{l-j} (D^{q-s,p,\alpha} \phi_k),
\end{aligned}$$

for  $0 \leq j \leq p' \leq m$  with

$$U_a^{(q,j,\alpha)} = \sum_{s=1}^q \sum_{l=1}^j \binom{q}{s} \binom{p+j}{l} D^{s,l,0} a D^{q-s,p+j-l,\alpha} \phi_k. \quad (49)$$

Hence,  $H_a^{(q,j,\alpha)}$ ,  $0 \leq j \leq p' - 1 < m$ , can be split as

$$H_a^{(q,p,\alpha)} = F_a^{(q,j,\alpha)} + S_a^{(q,j,\alpha)} + U_a^{(q,j,\alpha)},$$

with

$$\begin{aligned}
F_a^{(q,j,\alpha)} &= \sum_{s=2}^q \sum_{l=0}^{p-1} \frac{1}{l!} \binom{q}{s} D^{s,0,0} a D^{q-s+1,0,\alpha} \phi_k^{(l)} \rho^l \\
&\quad + \sum_{s=1}^q \sum_{l=j+1}^{p+j} \sum_{t=p+j-l}^{p-1} \frac{1}{(t-p-j+l)!} \binom{q}{s} \binom{p+j}{l} D^{s,l,\alpha} a D^{q-s,0,\alpha} \phi_k^{(t)} \rho^{t-p-j+l},
\end{aligned}$$

and

$$\begin{aligned}
S_a^{(q,j,\alpha)} &= \sum_{s=2}^q \binom{q}{s} D^{s,0,0} a J^p (D^{q-s+1,0,\alpha} \phi_k) \\
&\quad + \sum_{s=1}^q \sum_{l=j+1}^{p+j} \binom{q}{s} \binom{p+j}{l} D^{s,l,0} a J^{l-j} (D^{q-s,p,\alpha} \phi_k).
\end{aligned}$$

As in the discussion of section 5.1, the terms  $F_a^{(q,j,\alpha)}$ ,  $0 \leq j \leq p' \leq m$  can be calculated explicitly using the transport equations on  $\mathcal{I}$ . The terms  $S_a^{(q,j,\alpha)}$  are homogeneous in  $J^l(D^{i,n,\beta} \phi_k)$  with  $l \geq 1$ ,  $i + n + |\beta| \leq m$ ; hence they can be controlled by choosing a suitable  $\rho_*$ . The term  $U^{(q,j,\alpha)}$  contains  $\rho$ -derivatives of  $\phi_k$  up to order  $p + j - 1$ , accordingly it will be controlled using the induction hypothesis. A similar split can be introduced for the other  $H$ 's and  $K$ 's. Hence, one is



led to analyse the expression:

$$\begin{aligned}
0 = & \partial_\tau \left( (1 + \tau + a) |D^{q,j,\alpha}(\partial_\rho^p \phi_k)|^2 + (1 - \tau - a) |D^{q,j,\alpha}(\partial_\rho^p \phi_{k+1})|^2 \right) \\
& + \partial_\rho \left( (\rho + b) |D^{q,j,\alpha}(\partial_\rho^p \phi_{k+1})|^2 - (\rho + b) |D^{q,j,\alpha}(\partial_\rho^p \phi_k)|^2 \right) \\
& + (1 + c) \left( D^{q,j,\alpha}(\partial_\rho^p \bar{\phi}_k) D^{q,j,\alpha} X_+ (\partial_\rho^p \phi_{k+1}) + D^{q,j,\alpha}(\partial_\rho^p \phi_{k+1}) D^{q,j,\alpha} X_+ (\partial_\rho^p \bar{\phi}_k) \right) \\
& + (1 + c) \left( D^{q,j,\alpha}(\partial_\rho^p \phi_k) D^{q,j,\alpha} X_- (\partial_\rho^p \bar{\phi}_{k+1}) + D^{q,j,\alpha}(\partial_\rho^p \bar{\phi}_{k+1}) D^{q,j,\alpha} X_- (\partial_\rho^p \phi_k) \right) \\
& - 2 \left( p + j - q + k - 1 - f_k + \left( p - \frac{1}{2} \right) D^{0,1,0} b - \left( q - \frac{1}{2} \right) D^{1,0,0} a \right) |D^{q,j,\alpha}(\partial_\rho^p \phi_k)|^2 \\
& + 2 \left( p + j - q - k + g_k + \left( p - \frac{1}{2} \right) D^{0,1,0} b - \left( q - \frac{1}{2} \right) D^{1,0,0} a \right) |D^{q,j,\alpha}(\partial_\rho^p \phi_{k+1})|^2. \\
& + D^{q,j,\alpha}(\partial_\rho^p \bar{\phi}_k) \widehat{F}_k^{(q,j,\alpha)} + D^{q,j,\alpha}(\partial_\rho^p \phi_k) \overline{\widehat{F}}_k^{(q,j,\alpha)} + D^{q,j,\alpha}(\partial_\rho^p \bar{\phi}_k) \widehat{G}_k^{(q,j,\alpha)} + D^{q,j,\alpha}(\partial_\rho^p \phi_k) \overline{\widehat{G}}_k^{(q,j,\alpha)} \\
& + D^{q,j,\alpha}(\partial_\rho^p \bar{\phi}_k) \widehat{S}_k^{(q,j,\alpha)} + D^{q,j,\alpha}(\partial_\rho^p \phi_k) \overline{\widehat{S}}_k^{(q,j,\alpha)} + D^{q,j,\alpha}(\partial_\rho^p \bar{\phi}_k) \widehat{R}_k^{(q,j,\alpha)} + D^{q,j,\alpha}(\partial_\rho^p \phi_k) \overline{\widehat{R}}_k^{(q,j,\alpha)} \\
& + D^{q,j,\alpha}(\partial_\rho^p \bar{\phi}_k) \widehat{U}_k^{(q,j,\alpha)} + D^{q,j,\alpha}(\partial_\rho^p \phi_k) \overline{\widehat{U}}_k^{(q,j,\alpha)} + D^{q,j,\alpha}(\partial_\rho^p \bar{\phi}_k) \widehat{V}_k^{(q,j,\alpha)} + D^{q,j,\alpha}(\partial_\rho^p \phi_k) \overline{\widehat{V}}_k^{(q,j,\alpha)}, \tag{50}
\end{aligned}$$

with

$$\begin{aligned}
\widehat{F}_k^{(q,j,\alpha)} &= F_a^{(q,j,\alpha)} + F_b^{(q,j,\alpha)} + F_c^{(q,j,\alpha)} + F_{f_k}^{(q,j,\alpha)}, \\
\widehat{G}_k^{(q,j,\alpha)} &= G_a^{(q,j,\alpha)} + G_b^{(q,j,\alpha)} + G_c^{(q,j,\alpha)} + G_{g_k}^{(q,j,\alpha)}, \\
\widehat{R}_k^{(q,j,\alpha)} &= R_a^{(q,j,\alpha)} + R_b^{(q,j,\alpha)} + R_c^{(q,j,\alpha)} + R_{f_k}^{(q,j,\alpha)}, \\
\widehat{S}_k^{(q,j,\alpha)} &= S_a^{(q,j,\alpha)} + S_b^{(q,j,\alpha)} + S_c^{(q,j,\alpha)} + S_{g_k}^{(q,j,\alpha)}, \\
\widehat{U}_k^{(q,j,\alpha)} &= U_a^{(q,j,\alpha)} + U_b^{(q,j,\alpha)} + U_c^{(q,j,\alpha)} + U_{f_k}^{(q,j,\alpha)}, \\
\widehat{V}_k^{(q,j,\alpha)} &= V_a^{(q,j,\alpha)} + V_b^{(q,j,\alpha)} + V_c^{(q,j,\alpha)} + V_{g_k}^{(q,j,\alpha)},
\end{aligned}$$

for  $k = 0, 1$  and  $q + j + |\alpha| \leq m$ . The terms  $V_a^{(q,j,\alpha)}$ ,  $V_b^{(q,j,\alpha)}$ ,  $V_c^{(q,j,\alpha)}$ ,  $V_{g_k}^{(q,j,\alpha)}$  are defined in analogy to the expression for  $U_a^{(q,j,\alpha)}$  given by formulae (49) and contain  $\rho$ -derivatives of  $\phi_{k+1}$  up to order  $p + j - 1$ .

Following the ideas of the argument given in sections 4 and 5.1, integrating (50) over  $\mathcal{N}_t$  and

then using the Gauss theorem on the first two terms one obtains the inequality

$$\begin{aligned}
& \int_{\mathcal{S}_t} ((1+t+a)|D^{q,j,\alpha}(\partial_\rho^p \phi_k)|^2 + (1-t-a)|D^{q,j,\alpha}(\partial_\rho^p \phi_{k+1})|^2) d\rho d\mu \\
& + \int_{\mathcal{N}_t} (1+c) (D^{q,j,\alpha}(\partial_\rho^p \bar{\phi}_k) D^{q,j,\alpha} X_+(\partial_\rho^p \phi_{k+1}) + D^{q,j,\alpha}(\partial_\rho^p \phi_{k+1}) D^{q,j,\alpha} X_+(\partial_\rho^p \bar{\phi}_k)) d\tau d\rho d\mu \\
& + \int_{\mathcal{N}_t} (1+c) (D^{q,j,\alpha}(\partial_\rho^p \phi_k) D^{q,j,\alpha} X_-(\partial_\rho^p \bar{\phi}_{k+1}) + D^{q,j,\alpha}(\partial_\rho^p \bar{\phi}_{k+1}) D^{q,j,\alpha} X_-(\partial_\rho^p \phi_k)) d\tau d\rho d\mu \\
& + 2 \int_{\mathcal{N}_t} \left( p+j-q-k+g_k + \left(p-\frac{1}{2}\right) D^{0,1,0} b - \left(q-\frac{1}{2}\right) D^{1,0,0} a \right) |D^{q,j,\alpha}(\partial_\rho^p \phi_{k+1})|^2 d\tau d\rho d\mu \\
& + \int_{\mathcal{N}_t} \left( D^{q,j,\alpha}(\partial_\rho^p \bar{\phi}_k) \widehat{F}_k^{q,j,\alpha} + D^{q,j,\alpha}(\partial_\rho^p \phi_k) \overline{\widehat{F}}_k^{q,j,\alpha} \right) d\tau d\rho d\mu \\
& + \int_{\mathcal{N}_t} \left( D^{q,j,\alpha}(\partial_\rho^p \bar{\phi}_k) \widehat{G}_k^{q,j,\alpha} + D^{q,j,\alpha}(\partial_\rho^p \phi_k) \overline{\widehat{G}}_k^{q,j,\alpha} \right) d\tau d\rho d\mu \\
& + \int_{\mathcal{N}_t} \left( D^{q,j,\alpha}(\partial_\rho^p \bar{\phi}_k) \widehat{S}_k^{q,j,\alpha} + D^{q,j,\alpha}(\partial_\rho^p \phi_k) \overline{\widehat{S}}_k^{q,j,\alpha} \right) d\tau d\rho d\mu \\
& + \int_{\mathcal{N}_t} \left( D^{q,j,\alpha}(\partial_\rho^p \bar{\phi}_k) \widehat{R}_k^{q,j,\alpha} + D^{q,j,\alpha}(\partial_\rho^p \phi_k) \overline{\widehat{R}}_k^{q,j,\alpha} \right) d\tau d\rho d\mu \\
& + \int_{\mathcal{N}_t} \left( D^{q,j,\alpha}(\partial_\rho^p \bar{\phi}_k) \widehat{U}_k^{q,j,\alpha} + D^{q,j,\alpha}(\partial_\rho^p \phi_k) \overline{\widehat{U}}_k^{q,j,\alpha} \right) d\tau d\rho d\mu \\
& + \int_{\mathcal{N}_t} \left( D^{q,j,\alpha}(\partial_\rho^p \bar{\phi}_k) \widehat{V}_k^{q,j,\alpha} + D^{q,j,\alpha}(\partial_\rho^p \phi_k) \overline{\widehat{V}}_k^{q,j,\alpha} \right) d\tau d\rho d\mu \\
& \leq \int_{\mathcal{S}_0} (|D^{q,j,\alpha}(\partial_\rho^p \phi_k)|^2 + |D^{q,j,\alpha}(\partial_\rho^p \phi_{k+1})|^2) d\rho d\mu \\
& + 2 \int_{\mathcal{N}_t} \left( p+j-q+k-1-f_k + \left(p-\frac{1}{2}\right) D^{0,1,0} b - \left(q-\frac{1}{2}\right) D^{1,0,0} a \right) |D^{q,j,\alpha}(\partial_\rho^p \phi_k)|^2 d\tau d\rho d\mu,
\end{aligned}$$

for  $k = 0, 1$  and  $0 \leq j \leq p' \leq m$ . As in section 5.1, the following inequalities will be used:

$$\begin{aligned}
& \left| \int_{\mathcal{N}_t} \left( D^{q,j,\alpha}(\partial_\rho^p \bar{\phi}_k) \widehat{F}_k^{(q,j,\alpha)} + D^{q,j,\alpha}(\partial_\rho^p \phi_k) \overline{\widehat{F}}_k^{(q,j,\alpha)} \right) d\tau d\rho d\mu \right| \leq \int_{\mathcal{N}_t} \left( \varepsilon |D^{q,j,\alpha}(\partial_\rho^p \phi_k)|^2 + \frac{1}{\varepsilon} |\widehat{F}_k^{(q,j,\alpha)}|^2 \right) d\tau d\rho d\mu, \\
& \left| \int_{\mathcal{N}_t} \left( D^{q,j,\alpha}(\partial_\rho^p \bar{\phi}_k) \widehat{G}_k^{(q,j,\alpha)} + D^{q,j,\alpha}(\partial_\rho^p \phi_k) \overline{\widehat{G}}_k^{(q,j,\alpha)} \right) d\tau d\rho d\mu \right| \leq \int_{\mathcal{N}_t} \left( \varepsilon |D^{q,j,\alpha}(\partial_\rho^p \phi_k)|^2 + \frac{1}{\varepsilon} |\widehat{G}_k^{(q,j,\alpha)}|^2 \right) d\tau d\rho d\mu, \\
& \left| \int_{\mathcal{N}_t} \left( D^{q,j,\alpha}(\partial_\rho^p \bar{\phi}_k) \widehat{U}_k^{(q,j,\alpha)} + D^{q,j,\alpha}(\partial_\rho^p \phi_k) \overline{\widehat{U}}_k^{(q,j,\alpha)} \right) d\tau d\rho d\mu \right| \leq \int_{\mathcal{N}_t} \left( \zeta |D^{q,j,\alpha}(\partial_\rho^p \phi_k)|^2 + \frac{1}{\zeta} |\widehat{U}_k^{(q,j,\alpha)}|^2 \right) d\tau d\rho d\mu, \\
& \left| \int_{\mathcal{N}_t} \left( D^{q,j,\alpha}(\partial_\rho^p \bar{\phi}_k) \widehat{V}_k^{(q,j,\alpha)} + D^{q,j,\alpha}(\partial_\rho^p \phi_k) \overline{\widehat{V}}_k^{(q,j,\alpha)} \right) d\tau d\rho d\mu \right| \leq \int_{\mathcal{N}_t} \left( \zeta |D^{q,j,\alpha}(\partial_\rho^p \phi_k)|^2 + \frac{1}{\zeta} |\widehat{V}_k^{(q,j,\alpha)}|^2 \right) d\tau d\rho d\mu,
\end{aligned}$$

with  $\varepsilon, \zeta > 0$ . Furthermore, using the same arguments as in subsection 5.1 one has that there is a  $\rho_* > 0$  such that

$$|\widehat{S}_k^{(q,j,\alpha)}|^2 \leq \delta, \quad |\widehat{R}_k^{(q,j,\alpha)}|^2 \leq \delta,$$

for  $(\tau, \rho, \varsigma) \in \mathcal{N}_t$ ,  $t \in [0, 1]$ . Hence,

$$\begin{aligned}
& \left| \int_{\mathcal{N}_t} \left( D^{q,j,\alpha}(\partial_\rho^p \bar{\phi}_k) \widehat{S}_k^{(q,j,\alpha)} + D^{q,j,\alpha}(\partial_\rho^p \phi_k) \overline{\widehat{S}}_k^{(q,j,\alpha)} \right) d\tau d\rho d\mu \right| \leq 2\delta \int_{\mathcal{N}_t} (1 + |D^{q,j,\alpha}(\partial_\rho^p \phi_k)|^2) d\tau d\rho d\mu, \\
& \left| \int_{\mathcal{N}_t} \left( D^{q,j,\alpha}(\partial_\rho^p \bar{\phi}_{k+1}) \widehat{R}_k^{(q,j,\alpha)} + D^{q,j,\alpha}(\partial_\rho^p \phi_{k+1}) \overline{\widehat{R}}_k^{(q,j,\alpha)} \right) d\tau d\rho d\mu \right| \leq 2\delta \int_{\mathcal{N}_t} (1 + |D^{q,j,\alpha}(\partial_\rho^p \phi_{k+1})|^2) d\tau d\rho d\mu.
\end{aligned}$$

Using the above estimates one obtains

$$\begin{aligned}
& \int_{\mathcal{S}_t} \left( (1+t+a) |D^{q,j,\alpha}(\partial_\rho^p \phi_k)|^2 + (1-t-a) |D^{q,j,\alpha} \partial_\rho^p(\phi_{k+1})|^2 \right) d\rho d\mu \\
& + \int_{\mathcal{N}_t} (1+c) \left( D^{q,j,\alpha}(\partial_\rho^p \bar{\phi}_k) D^{q,j,\alpha} X_+(\partial_\rho^p \phi_{k+1}) + D^{q,j,\alpha}(\partial_\rho^p \phi_{k+1}) D^{q,j,\alpha} X_+(\partial_\rho^p \bar{\phi}_k) \right) d\tau d\rho d\mu \\
& + \int_{\mathcal{N}_t} (1+c) \left( D^{q,j,\alpha}(\partial_\rho^p \phi_k) D^{q,j,\alpha} X_-(\partial_\rho^p \bar{\phi}_{k+1}) + D^{q,j,\alpha}(\partial_\rho^p \bar{\phi}_{k+1}) D^{q,j,\alpha} X_-(\partial_\rho^p \phi_k) \right) d\tau d\rho d\mu \\
& + 2 \int_{\mathcal{N}_t} \left( p+j-q-k-\frac{\varepsilon}{2}-\delta-\frac{\zeta}{2}+g_k+\left(p-\frac{1}{2}\right) D^{0,1,0}b-\left(q-\frac{1}{2}\right) D^{1,0,0}a \right) |D^{q,j,\alpha}(\partial_\rho^p \phi_{k+1})|^2 d\tau d\rho d\mu \\
& \leq \int_{\mathcal{S}_0} (|D^{q,j,\alpha}(\partial_\rho^p \phi_k)|^2 + |D^{q,j,\alpha}(\partial_\rho^p \phi_{k+1})|^2) d\rho d\mu \\
& + 2 \int_{\mathcal{N}_t} \left( p+j-q+k-1-f_k+\frac{\varepsilon}{2}+\delta+\frac{\zeta}{2}+\left(p-\frac{1}{2}\right) D^{0,1,0}b \right. \\
& \quad \left. -\left(q-\frac{1}{2}\right) D^{1,0,0}a \right) |D^{q,j,\alpha}(\partial_\rho^p \phi_k)|^2 d\tau d\rho d\mu \\
& + \int_{\mathcal{N}_t} \left( 2\delta+\frac{1}{\varepsilon} |\widehat{F}_k^{(q,j,\alpha)}|^2 + \frac{1}{\varepsilon} |\widehat{G}_k^{(q,j,\alpha)}|^2 + \frac{1}{\zeta} |\widehat{U}_k^{(q,j,\alpha)}|^2 + \frac{1}{\zeta} |\widehat{V}_k^{(q,j,\alpha)}|^2 \right) d\tau d\rho d\mu,
\end{aligned}$$

for a suitable  $\rho_* > 0$ . As in section 5.1, it is used that  $f_k, g_k, D^{1,0,0}a$  and  $D^{0,1,0}b$  are all  $\mathcal{O}(\rho)$ . Hence as in the case  $j = 0$  one can set

$$\begin{aligned}
& \left| \left( p-\frac{1}{2} \right) D^{0,1,0}b - \left( q-\frac{1}{2} \right) D^{1,0,0}a + g_k \right| \leq \eta, \\
& \left| \left( p-\frac{1}{2} \right) D^{0,1,0}b - \left( q-\frac{1}{2} \right) D^{1,0,0}a - f_k \right| \leq \eta,
\end{aligned}$$

with a suitably small choice of  $\rho_* > 0$ . Putting everything together one arrives at the inequality

$$\begin{aligned}
& \int_{\mathcal{S}_t} \left( (1+t+a) |D^{q,j,\alpha}(\partial_\rho^p \phi_k)|^2 + (1-t-a) |D^{q,j,\alpha} \partial_\rho^p(\phi_{k+1})|^2 \right) d\rho d\mu \\
& + \int_{\mathcal{N}_t} (1+c) \left( D^{q,j,\alpha}(\partial_\rho^p \bar{\phi}_k) D^{q,j,\alpha} X_+(\partial_\rho^p \phi_{k+1}) + D^{q,j,\alpha}(\partial_\rho^p \phi_{k+1}) D^{q,j,\alpha} X_+(\partial_\rho^p \bar{\phi}_k) \right) d\tau d\rho d\mu \\
& + \int_{\mathcal{N}_t} (1+c) \left( D^{q,j,\alpha}(\partial_\rho^p \phi_k) D^{q,j,\alpha} X_-(\partial_\rho^p \bar{\phi}_{k+1}) + D^{q,j,\alpha}(\partial_\rho^p \bar{\phi}_{k+1}) D^{q,j,\alpha} X_-(\partial_\rho^p \phi_k) \right) d\tau d\rho d\mu \\
& + 2 \left( p+j-q-k-\frac{\varepsilon}{2}-\delta-\frac{\zeta}{2}-\eta \right) \int_{\mathcal{N}_t} |D^{q,j,\alpha}(\partial_\rho^p \phi_{k+1})|^2 d\tau d\rho d\mu \\
& \leq \int_{\mathcal{S}_0} (|D^{q,j,\alpha}(\partial_\rho^p \phi_k)|^2 + |D^{q,j,\alpha}(\partial_\rho^p \phi_{k+1})|^2) d\rho d\mu \\
& + 2 \left( p+j-q+k-1+\frac{\varepsilon}{2}+\delta+\frac{\zeta}{2}+\eta \right) \int_{\mathcal{N}_t} |D^{q,j,\alpha}(\partial_\rho^p \phi_k)|^2 d\tau d\rho d\mu \\
& + \int_{\mathcal{N}_t} \left( 2\delta+\frac{1}{\varepsilon} |\widehat{F}_k^{(q,j,\alpha)}|^2 + \frac{1}{\varepsilon} |\widehat{G}_k^{(q,j,\alpha)}|^2 + \frac{1}{\zeta} |\widehat{U}_k^{(q,j,\alpha)}|^2 + \frac{1}{\zeta} |\widehat{V}_k^{(q,j,\alpha)}|^2 \right) d\tau d\rho d\mu,
\end{aligned} \tag{51}$$

for suitably small  $\rho_*$  and  $k = 0, 1$ . Summing the inequality (51) over all admissible values of  $q, j$  and  $\alpha$  for  $q+j+|\alpha| \leq m$  and  $0 \leq j < p'$  one sees that the terms involving the second and third integral of the left hand side of the last expression vanish by virtue of lemma 1 in appendix B.

Accordingly one has

$$\begin{aligned}
& \sum_{\substack{q+j+|\alpha|\leq m \\ 0\leq j\leq p'}} \int_{\mathcal{S}_t} \left( (1+t+a)|D^{q,j,\alpha}(\partial_\rho^p \phi_k)|^2 + (1-t-a)|D^{q,j,\alpha} \partial_\rho^p(\phi_{k+1})|^2 \right) d\rho d\mu \\
& + 2 \sum_{\substack{q+j+|\alpha|\leq m \\ 0\leq j\leq p'}} \left( p+j-q-k-\frac{\varepsilon}{2}-\delta-\frac{\zeta}{2}-\eta \right) \int_{\mathcal{N}_t} |D^{q,j,\alpha}(\partial_\rho^p \phi_{k+1})|^2 d\tau d\rho d\mu \\
& \leq \sum_{\substack{q+j+|\alpha|\leq m \\ 0\leq j\leq p'}} \int_{\mathcal{S}_0} (|D^{q,j,\alpha}(\partial_\rho^p \phi_k)|^2 + |D^{q,j,\alpha}(\partial_\rho^p \phi_{k+1})|^2) d\rho d\mu \\
& + 2 \sum_{\substack{q+j+|\alpha|\leq m \\ 0\leq j\leq p'}} \left( p+j-q+k-1+\frac{\varepsilon}{2}+\delta+\frac{\zeta}{2}+\eta \right) \int_{\mathcal{N}_t} |D^{q,j,\alpha}(\partial_\rho^p \phi_k)|^2 d\tau d\rho d\mu \\
& + \sum_{\substack{q+j+|\alpha|\leq m \\ 0\leq j\leq p'}} \int_{\mathcal{N}_t} \left( 2\delta + \frac{1}{\varepsilon} |\widehat{F}_k^{(q,j,\alpha)}|^2 + \frac{1}{\varepsilon} |\widehat{G}_k^{(q,j,\alpha)}|^2 + \frac{1}{\zeta} |\widehat{U}_k^{(q,j,\alpha)}|^2 + \frac{1}{\zeta} |\widehat{V}_k^{(q,j,\alpha)}|^2 \right) d\tau d\rho d\mu,
\end{aligned} \tag{52}$$

for suitable  $\rho_*$ ,  $k = 0, 1$ . Now, noting that

$$\begin{aligned}
& \sum_{\substack{q+j+|\alpha|\leq m \\ 0\leq j\leq p'}} \left( p+j-q-k-\frac{\varepsilon}{2}-\delta-\frac{\zeta}{2}-\eta \right) \int_{\mathcal{N}_t} |D^{q,j,\alpha}(\partial_\rho^p \phi_{k+1})|^2 d\tau d\rho d\mu \\
& \geq \left( p-m-1-\frac{\varepsilon}{2}-\delta-\frac{\zeta}{2}-\eta \right) \sum_{\substack{q+j+|\alpha|\leq m \\ 0\leq j\leq p'}} \int_{\mathcal{N}_t} |D^{q,j,\alpha}(\partial_\rho^p \phi_{k+1})|^2 d\tau d\rho d\mu, \\
& \sum_{\substack{q+j+|\alpha|\leq m \\ 0\leq j\leq p'}} \left( p+j-q+k-1+\frac{\varepsilon}{2}+\delta+\frac{\zeta}{2}+\eta \right) \int_{\mathcal{N}_t} |D^{q,j,\alpha}(\partial_\rho^p \phi_k)|^2 d\tau d\rho d\mu \\
& \leq \left( p+m+\frac{\varepsilon}{2}+\delta+\frac{\zeta}{2}+\eta \right) \sum_{\substack{q+j+|\alpha|\leq m \\ 0\leq j\leq p'}} \int_{\mathcal{N}_t} |D^{q,j,\alpha}(\partial_\rho^p \phi_k)|^2 d\tau d\rho d\mu,
\end{aligned}$$

one obtains from (52) the basic inequality

$$\begin{aligned}
& \sum_{\substack{q+j+|\alpha|\leq m \\ 0\leq j\leq p'}} \int_{\mathcal{S}_t} \left( (1+t+a)|D^{q,j,\alpha}(\partial_\rho^p \phi_k)|^2 + (1-t-a)|D^{q,j,\alpha} \partial_\rho^p(\phi_{k+1})|^2 \right) d\rho d\mu \\
& + 2 \left( p-m-1-\frac{\varepsilon}{2}-\delta-\frac{\zeta}{2}-\eta \right) \sum_{\substack{q+j+|\alpha|\leq m \\ 0\leq j\leq p'}} \int_{\mathcal{N}_t} |D^{q,j,\alpha}(\partial_\rho^p \phi_{k+1})|^2 d\tau d\rho d\mu \\
& \leq \sum_{\substack{q+j+|\alpha|\leq m \\ 0\leq j\leq p'}} \int_{\mathcal{S}_0} (|D^{q,j,\alpha}(\partial_\rho^p \phi_k)|^2 + |D^{q,j,\alpha}(\partial_\rho^p \phi_{k+1})|^2) d\rho d\mu \\
& + 2 \left( p+m+\frac{\varepsilon}{2}+\delta+\frac{\zeta}{2}+\eta \right) \sum_{\substack{q+j+|\alpha|\leq m \\ 0\leq j\leq p'}} \int_{\mathcal{N}_t} |D^{q,j,\alpha}(\partial_\rho^p \phi_k)|^2 d\tau d\rho d\mu \\
& + \sum_{\substack{q+j+|\alpha|\leq m \\ 0\leq j\leq p'}} \int_{\mathcal{N}_t} \left( 2\delta + \frac{1}{\varepsilon} |\widehat{F}_k^{(q,j,\alpha)}|^2 + \frac{1}{\varepsilon} |\widehat{G}_k^{(q,j,\alpha)}|^2 + \frac{1}{\zeta} |\widehat{U}_k^{(q,j,\alpha)}|^2 + \frac{1}{\zeta} |\widehat{V}_k^{(q,j,\alpha)}|^2 \right) d\tau d\rho d\mu,
\end{aligned} \tag{53}$$

for a suitably small  $\rho_* > 0$  and  $k = 0, 1$ . As in the discussion of the preceding sections, the integers  $p$  and  $m$  are to be chosen such that the second term in inequality (53) is positive. That

is, one requires

$$p > m + 1 + \frac{\varepsilon}{2} + \delta + \frac{\zeta}{2} + \eta.$$

The constants  $\varepsilon, \delta, \zeta, \eta$  can be chosen such that

$$\frac{\varepsilon}{2} + \delta + \frac{\zeta}{2} + \eta < 1,$$

by means of a suitably small choice of  $\rho_*$ . Note that the “shrinking” of  $\rho_*$  is only performed a finite number of times, hence the neighbourhood around spatial infinity does not collapse to a point. Using the same arguments leading to inequality (44) one obtains from (53) that

$$\begin{aligned} & \sum_{\substack{q+j+|\alpha| \leq m \\ 0 \leq j \leq p'}} \int_{\mathcal{S}_t} |D^{q,j,\alpha}(\partial_\rho^p \phi_k)|^2 d\rho d\mu \\ & \leq \sum_{\substack{q+j+|\alpha| \leq m \\ 0 \leq j \leq p'}} \int_{\mathcal{S}_0} (|D^{q,j,\alpha}(\partial_\rho^p \phi_k)|^2 + |D^{q,j,\alpha}(\partial_\rho^p \phi_{k+1})|^2) d\rho d\mu \\ & \quad + 2(p+m+1) \sum_{\substack{q+j+|\alpha| \leq m \\ 0 \leq j \leq p'}} \int_{\mathcal{N}_t} |D^{q,j,\alpha}(\partial_\rho^p \phi_k)|^2 d\tau d\rho d\mu \\ & \quad + \sum_{\substack{q+j+|\alpha| \leq m \\ 0 \leq j \leq p'}} \int_{\mathcal{N}_t} \left( 2\delta + \frac{1}{\varepsilon} |\widehat{F}_k^{(q,j,\alpha)}|^2 + \frac{1}{\varepsilon} |\widehat{G}_k^{(q,j,\alpha)}|^2 + \frac{1}{\zeta} |\widehat{U}_k^{(q,j,\alpha)}|^2 + \frac{1}{\zeta} |\widehat{V}_k^{(q,j,\alpha)}|^2 \right) d\tau d\rho d\mu, \end{aligned} \tag{54}$$

valid for  $t \in [0, 1]$ ,  $p \geq m+2$ ,  $k = 0, 1$ , and suitably small  $0 < \rho_* < a_*$ . The companion inequality for  $\phi_2$  is

$$\begin{aligned} & 2(p-m-2) \sum_{\substack{q+j+|\alpha| \leq m \\ 0 \leq j \leq p'}} \int_{\mathcal{N}_t} |D^{q,j,\alpha}(\partial_\rho^p \phi_2)|^2 d\tau d\rho d\mu \\ & \leq \sum_{\substack{q+j+|\alpha| \leq m \\ 0 \leq j \leq p'}} \int_{\mathcal{S}_0} (|D^{q,j,\alpha}(\partial_\rho^p \phi_1)|^2 + |D^{q,j,\alpha}(\partial_\rho^p \phi_2)|^2) d\rho d\mu \\ & \quad + 2(p+m+1) \sum_{\substack{q+j+|\alpha| \leq m \\ 0 \leq j \leq p'}} \int_{\mathcal{N}_t} |D^{q,j,\alpha}(\partial_\rho^p \phi_1)|^2 d\tau d\rho d\mu \\ & \quad + \sum_{\substack{q+j+|\alpha| \leq m \\ 0 \leq j \leq p'}} \int_{\mathcal{N}_t} \left( 2\delta + \frac{1}{\varepsilon} |\widehat{F}_1^{(q,j,\alpha)}|^2 + \frac{1}{\varepsilon} |\widehat{G}_1^{(q,j,\alpha)}|^2 + \frac{1}{\zeta} |\widehat{U}_1^{(q,j,\alpha)}|^2 + \frac{1}{\zeta} |\widehat{V}_1^{(q,j,\alpha)}|^2 \right) d\tau d\rho d\mu, \end{aligned} \tag{55}$$

for  $t \in [0, 1]$ ,  $p \geq m+2$  and suitably small  $\rho_*$ . Gronwall’s argument applied to (54) yields

$$\begin{aligned} & \sum_{\substack{q+j+|\alpha| \leq m \\ 0 \leq j \leq p'}} \int_{\mathcal{N}_t} |D^{q,j,\alpha}(\partial_\rho^p \phi_k)|^2 d\tau d\rho d\mu \\ & \leq \frac{1}{2(p+m+1)} \left( e^{2(p+m+1)t} - 1 \right) \sum_{\substack{q+j+|\alpha| \leq m \\ 0 \leq j \leq p'}} \int_{\mathcal{S}_0} (|D^{q,j,\alpha}(\partial_\rho^p \phi_k)|^2 + |D^{q,j,\alpha}(\partial_\rho^p \phi_{k+1})|^2) d\rho d\mu \\ & \quad + e^{2(p+m+1)t} \int_0^t e^{-2(p+m+1)s} \left( \sum_{\substack{q+j+|\alpha| \leq m \\ 0 \leq j \leq p'}} \int_{\mathcal{N}_t} \left( 2\delta + \frac{1}{\varepsilon} |\widehat{F}_k^{(q,j,\alpha)}|^2 + \frac{1}{\varepsilon} |\widehat{G}_k^{(q,j,\alpha)}|^2 + \frac{1}{\zeta} |\widehat{U}_k^{(q,j,\alpha)}|^2 + \frac{1}{\zeta} |\widehat{V}_k^{(q,j,\alpha)}|^2 \right) d\tau d\rho d\mu \right) ds, \end{aligned} \tag{56}$$

for  $k = 0, 1$ ,  $t \in [0, 1]$  and suitably small  $0 < \rho_* < a_*$ . To conclude the argument it is noted that because of the induction hypothesis, there are constants  $\hat{C}_1(\rho_*) > 0$  and  $\hat{C}_2(\rho_*) > 0$ , such that

$$\begin{aligned} & \sum_{\substack{q+j+|\alpha| \leq m \\ 0 \leq j \leq p'}} \int_{\mathcal{N}_t} \left( |\hat{U}_k^{(q,j,\alpha)}|^2 + |\hat{V}_k^{(q,j,\alpha)}|^2 \right) d\rho d\mu \\ & \leq \hat{C}_1(\rho_*) \sum_{k=0}^2 \int_{\mathcal{S}_0} \left( \sum_{\substack{q+j+|\alpha| \leq m \\ 0 \leq j \leq p'}} |D^{(q,j,\alpha)}(\partial_\rho^p \phi_k)|^2 \right) d\tau d\rho d\mu + \hat{C}_2(\rho_*). \end{aligned} \quad (57)$$

Furthermore, if the solutions, to the transport equations on  $\mathcal{I}$ ,  $\phi_k^{(l)}$  for  $k = 0, 1, 2$  and  $0 \leq l \leq p$ , are suitably smooth—and this can be controlled by a suitable choice of the initial data—then there is a constant  $\hat{D}(\rho_*) > 0$  such that

$$\sum_{\substack{q+j+|\alpha| \leq m \\ 0 \leq j \leq p'}} \int_{\mathcal{N}_t} \left( |\hat{F}_k^{(q,j,\alpha)}|^2 + |\hat{G}_k^{(q,j,\alpha)}|^2 \right) d\tau d\rho d\mu \leq \hat{D}(\rho_*). \quad (58)$$

Thus, combining the inequalities (54) and (55)—following the model of section 5.1—and using (57) and (58) one finds that there exist constants  $\tilde{C}_1(\rho_*) > 0$  and  $\tilde{C}_2(\rho_*) > 0$  such that

$$\begin{aligned} & \int_{\mathcal{N}_t} \sum_{\substack{q+j+|\alpha| \leq m \\ 0 \leq j \leq p'}} |D^{q,j,\alpha}(\partial_\rho^p \phi_k)|^2 d\tau d\rho d\mu \\ & \leq \tilde{C}_1(\rho_*) \sum_{k=0}^2 \int_{\mathcal{S}_0} \left( \sum_{\substack{q+j+|\alpha| \leq m \\ 0 \leq j \leq p'}} |D^{q,j,\alpha}(\partial_\rho^p \phi_k)|^2 \right) d\rho d\mu + \tilde{C}_2(\rho_*), \end{aligned}$$

for suitably small  $\rho_* > 0$  and  $p > m + 2$ . This concludes the induction argument.

## 6 Conclusions

Sections 5.1 and 5.2 have been concerned with the construction of  $L^2$ -type estimates for the quantities  $\partial_\rho^p \phi_k$ ,  $k = 0, 1, 2$  in the case of a Maxwell field propagating as a test field on a Schwarzschild background. The basic assumptions and main results of this construction are summarised in the following theorem.

**Theorem 1.** *Let  $m > 0$  an integer and  $\phi_k$ ,  $k = 0, 1, 2$  solutions to the Maxwell equations on a Schwarzschild background, equations (15a) to (15d). Furthermore, assume  $\phi_k$   $k = 0, 1, 2$  are of the form*

$$\phi_k = \sum_{l=|1-k|}^{p-1} \frac{1}{l!} \phi_k^{(l)} \rho^l + J^p(\partial_\rho^p \phi_k),$$

*with  $\phi_k^{(l)}$ ,  $0 \leq l \leq p-1$  solutions of the transport equations implied by the Maxwell equations on  $\mathcal{I}$ . Assume that the initial data for the equations (15a) to (15d) are such that the functions  $\phi_k^{(l)}$  are suitably smooth—so that the functions  $\hat{F}_k^{(q,j,\alpha)}$  and  $\hat{G}_k^{(q,j,\alpha)}$  are bounded in a neighbourhood of  $\mathcal{I}$ . Then, given  $p > m + 2$  there exist  $0 < \rho_* < a_*$  and constants  $C_1(\rho_*)$ ,  $C_2(\rho_*)$  such that*

$$\begin{aligned} & \int_{\mathcal{N}_t} \sum_{q+j+|\alpha| \leq m} |D^{q,j,\alpha}(\partial_\rho^p \phi_k)|^2 d\tau d\rho d\mu \\ & \leq C_1(\rho_*) \sum_{k=0}^2 \int_{\mathcal{S}_0} \left( \sum_{q+j+|\alpha| \leq m} |D^{q,j,\alpha}(\partial_\rho^p \phi_k)|^2 \right) d\rho d\mu + C_2(\rho_*), \end{aligned}$$

*for  $t \in [0, 1]$ .*

**Remark.** The precise nature of the regularity of the solutions to the transport equations on  $\mathcal{I}$  must be analysed on a case by case basis, and can be obtained by means of a careful reading of the arguments presented in this article. Their precise formulation goes beyond the scope of the present work. A translation of the conditions in terms of initial data can be obtained by means of the techniques of [15].

To conclude, assuming that one has a solution of the Maxwell equations on a Schwarzschild background of the form (18), then the estimates in theorem 1 allow to control the regularity of the remainder in the expansion in a similar way as it was done in section 4 for a flat background, at least for a small (but finite) neighbourhood of the cylinder at spatial infinity,  $\mathcal{I}$ , which includes the critical set,  $\mathcal{I}^+$ , and future null infinity,  $\mathcal{J}^+$ . The open question is now how to ensure the existence of solutions to the Maxwell equations of the desired form.

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## A Some spinors

Let  $\{o_A, \iota_A\}$  denote a normalised spinor dyad,  $\epsilon_{AB} o^A \iota^B = 1$ , where  $\epsilon_{AB}$  is the standard alternating spinor. The following spinors have been used:

$$\tau_{AA'} = o_A o_{A'} + \iota_A \iota_{A'}, \quad (59a)$$

$$\epsilon_{AB}^0 = o_A o_B, \quad \epsilon_{AB}^1 = o_{(A} \iota_{B)}, \quad \epsilon_{AB}^2 = \iota_A \iota_B, \quad (59b)$$

$$x_{AB} = \frac{1}{\sqrt{2}}(o_A \iota_B + \iota_A o_B), \quad y_{AB} = -\frac{1}{\sqrt{2}}\iota_A \iota_B, \quad z_{AB} = \frac{1}{\sqrt{2}}o_A o_B, \quad (59c)$$

$$\epsilon_{ABCD}^0 = o_A o_B o_C o_D, \quad \epsilon_{ABCD}^1 = o_{(A} o_B o_C \iota_{D)}, \quad \dots, \quad \epsilon_{ABCD}^4 = \iota_A \iota_B \iota_C \iota_D, \quad (59d)$$

$$h_{ABCD} = -\epsilon_{A(C} \epsilon_{D)B}. \quad (59e)$$

## B Some commutators on $SU(2)$

Consider the basis

$$u_1 = \frac{1}{2} \begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix}, \quad u_2 = \frac{1}{2} \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}, \quad u_3 = \frac{1}{2} \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix},$$

of the Lie algebra  $su(2)$ , with  $u_3$  the generator of the subgroup  $U(1)$ . Let  $Z_i$ ,  $i = 1, 2, 3$  denote the (real) left invariant vector fields generated by  $u_i$  on the Lie group  $SU(2)$ . They satisfy the following commutator relations

$$[Z_1, Z_2] = Z_3, \quad [Z_1, Z_3] = -Z_2, \quad [Z_2, Z_3] = Z_1. \quad (60)$$

We also make use of the following combinations of the  $Z_i$

$$X_+ = -Z_2 - iZ_1, \quad X_- = -Z_2 + iZ_1, \quad X = -2iZ_3. \quad (61)$$

Their commutators are

$$[X, X_+] = 2X_+, \quad [X, X_-] = -2X_-, \quad [X_+, X_-] = -X. \quad (62)$$

As in the main text let  $Z^\alpha = Z_1^{\alpha_1} Z_2^{\alpha_2} Z_3^{\alpha_3}$ . The operators provide a basis for the universal enveloping algebra of  $su(2)$ . The following result of [4] has been used several times in the main text.

**Lemma 1.** *For any smooth complex-valued functions  $f, g$  on  $SU(2)$  the operators  $Z^\alpha$  satisfy*

$$\sum_{|\alpha| \leq m} \int_{SU(2)} (Z^\alpha X_\pm f Z^\alpha g + Z^\alpha f Z^\alpha X_\pm g) d\mu = 0, \quad (63)$$

where  $d\mu$  denotes the normalised Haar measure on  $SU(2)$ .

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